NCERT Solutions for Class 12 Maths Chapter 7 – Integrals

Exercise 7.1

1.

Find an anti-derivative (or integral) of the following functions by the method of inspection.

$$\sin 2x$$

Ans - We use the method of inspection as follows.

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x \Rightarrow -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

$\therefore$ Anti-derivative of sin 2z is $-\frac{1}{2}\cos 2x$

2.

Find an anti-derivative (or integral) of the following functions by the method of inspection

$$\cos 3x$$

Ans - We use the method of inspection as follows

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x \Rightarrow \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

$\therefore$ Anti - derivative of cos 3x is $\frac{1}{3}\sin 3x$

3.

Find an anti-derivative (or integral) of the following functions by the method of inspection

$$e^{2x}$$

Ans - We use the method of inspection as follows

$$\frac{d}{dx}(e^{2x}) \Rightarrow 2e^{2x} = \frac{1}{2}\frac{d}{dx}(e^{2x})$$

$$\because e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

$\therefore$ Anti-derivative of e^{2x} is $\frac{1}{2}e^{2x}$

4.

Find an anti-derivative (or integral) of the following functions by the method of inspection.

$$(ax + b)^2$$

Ans - We use the method of inspection as follows

$$\frac{d}{dx}(ax + b)^3 = 3a(ax + b)^2$$

$$(ax + b)^2 = \frac{1}{3a}\frac{d}{dx}(ax + b)^3$$

$$\therefore (ax + b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax + b)^3\right)$$

$\therefore$ Anti $-$ derivative of $(ax + b)^2$ is $(ax + b)^3$.

5.

Find an anti-derivative (or integral) of the following functions by the method of inspection.

$$\sin 2x - 4e^{3x}$$

Ans - We use the method of inspection as follows

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = (\sin 2x - 4e^{3x})$$

∴ Anti-derivative of $(\sin 2x - 4e^{3x})$ is $\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right)$

6.

Find the following integral

$$\int (4e^{\alpha} + 1)dx$$

Ans $- \int (4e^{3x} + 1)dx$

$$= 4\int e^{3x}dx + \int 1dx$$

$$= 4\left(\frac{e^{3x}}{3}\right) + x + C$$

$$= \frac{4}{3}e^{3x} + x + C$$

7.

Find the following integral

$$\int x^2\left(1 - \frac{1}{x^2}\right)dx$$

Ans - $\int x^2\left(1 - \frac{1}{x^2}\right)dx$

$$= \int (x^2 - 1)dx$$

$$= \frac{x^1}{3} - x + C$$

8.

Find the following integral

$$\int (ax^2 + bx + c)dx$$

Ans - $\int (ax^2 + bx + c)dx$

$= a\int x^2 dx + b\int xdx + c\int 1.dx$

$= a\left(\dfrac{x^3}{3}\right) + b\left(\dfrac{x^2}{2}\right) + cx + D$

$= \dfrac{ax^1}{3} + \dfrac{bx^2}{2} + cx + D$

9.

Find the following integral

$\int (2x^2 + e^x)dx$

Ans - $\int (2x^2 + e^x)dx$

$= 2\int x^2 dx + \int e^x dx$

10.

Find the following integral

$$\int \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2 dx$$

Ans - $\int \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2 dx$

$= \int \left(x + \dfrac{1}{x} - 2\right)dx$

$= \int xdx + \int \dfrac{1}{x}dx - 2\int 1.dx$

$= \dfrac{x^2}{2} + \log|x| - 2x + C$

11.

Find the following integral

$$\int \frac{x^3 + 5x^2 - 4}{x^2} \, dx$$

Ans - $\int \frac{x^3 + 5x^2 - 4}{x^2} \, dx$

$= \int (x + 5 - 4x^{-2}) \, dx$

$= \int x \, dx + 5 \int 1. \, dx - 4 \int x^{-2} \, dx$

$= \frac{x^2}{2} + 5x + \frac{4}{x} + C$

12.

Find the following integral

$$\int \frac{x^3 + 3x + 4}{\sqrt{x}} \, dx$$

Ans $- \int \frac{x^3 + 3x + 4}{\sqrt{x}} \, dx$

$= \int \left(x^{\frac{5}{2}} + 3x^{\frac{1}{2}} + 4x^{-\frac{1}{2}} \right) dx$

$= \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}} + \frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}} + C$

13.

Find the following integral

$$\int \frac{x^3 - x^2 + x - 1}{x - 1} \, dx$$

$$\text{Ans} - \int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$$

$$= \int \frac{x^2(x - 1) + x - 1}{x - 1}\, dx$$

$$= \int \frac{(x - 1)(x^2 + 1)}{x - 1}$$

We obtain, on dividing:

$$= \int (x^2 + 1)\, dx$$

$$= \int x^2\, dx + \int 1.\, dx$$

$$= \frac{x^3}{3} + x + C$$

14.

Find the following integral

$$\int (1 - x)\sqrt{x}\, dx$$

Ans - $\int (1 - x)\sqrt{x}\, dx$

$$= \int \left(\sqrt{x} - x^{\frac{3}{2}}\right) dx$$

$$= \int x^{\frac{1}{2}}\, dx - \int x^{\frac{3}{2}}\, dx$$

$$= \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + C$$

15.

Find the following integral

$$\int \sqrt{x}(3x^2 + 2x + 3)\, dx$$

Ans - $\int \sqrt{x}(3x^2 + 2x + 3)dx$

$$= 3\int \left(x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}} \right)$$

$$= 3\int x^{\frac{5}{2}}dx + 2\int x^{\frac{3}{2}}dx + 3\int x^{\frac{1}{2}}dx$$

$$= \frac{6}{7}x^{\frac{7}{2}} + \frac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$$

16.

Find the following integral

$$\int (2x - 3\cos x + e^x)dx$$

Ans $- \int (2x - 3\cos x + e^x)dx$

$$= 2\int xdx - 3\int \cos xdx + \int e^x dx$$

$$= \frac{2x^2}{2} - 3(\sin x) + e^x + C$$

$$= x^2 - 3\sin x + e^x + C$$

17.

Find the following integral

$$\int (2x^2 - 3\sin x + 5\sqrt{x})dx$$

Ans - $\int (2x^2 - 3\sin x + 5\sqrt{x})dx$

$$= 2\int x^2 dx - 3\int \sin xdx + 5\int x^{\frac{1}{2}}dx$$

$$= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{\frac{3}{2}} + C$$

18.

Find the following integral

$$\int secx(secx + tanx)dx$$

Ans - $\int secx(secx + tanx)dx$

$$= \int (sec^2x + secxtanx)dx$$

$$= \int sec^2xdx + \int secxtanxdx$$

$$= tanx + secx + C$$

19.

Find the following integral

$$\int \frac{sec^2x}{cosec^2x}dx$$

Ans - $\int \frac{sec^2x}{cosec^2x}dx$

$$= \int \frac{\frac{1}{cos^2x}}{\frac{1}{sin^2x}}dx$$

$$= \int \frac{sin^2x}{cos^2x}dx$$

$$= \int tan^2xdx$$

$$= \int sec^2xdx - \int 1dx$$

$$= tanx - x + C$$

20.

Find the following integral

$$\int \frac{2 - 3\sin x}{\cos^2 x}\, dx$$

Ans - $\int \dfrac{2 - 3\sin x}{\cos^2 x}\, dx$

$$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$$

$$= \int 2\sec^2 x\, dx - 3 \int \tan x \sec x\, dx$$

$$= 2\tan x - 3\sec x + C$$

21.

The anti - derivative of $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}} \right)$ equals

A. $\dfrac{1}{3} x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ 　　 B. $\dfrac{2}{3} x^{\frac{2}{3}} + \dfrac{1}{2} x^2 + C$

C. $\dfrac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ 　　 D. $\dfrac{3}{2} x^{\frac{3}{2}} + \dfrac{1}{2} x^{\frac{1}{2}} + C$

Ans -

$$\left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) = \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

$\therefore$ Correct answer is C.

22.

If $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$ then $f(x)$ is

A. $x^4 + \frac{1}{x^3} - \frac{129}{8}$

B. $x^3 + \frac{1}{x^4} + \frac{129}{8}$

C. $x^4 + \frac{1}{x^3} + \frac{129}{8}$

D. $x^3 + \frac{1}{x^4} - \frac{129}{8}$

Ans - Given, $\frac{d}{dx}f(x) = 4x^3 - \frac{3}{x^4}$

Anti derivative of $4x^3 - \frac{3}{x^4}$

$\therefore f(x) = \int 4x^3 - \frac{3}{x^4} = f(x)$

$f(x) = 4\int x^3 dx - 3\int (x^{-4})dx$

$f(x) = 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C$

$f(x) = x^4 + \frac{1}{x^3} + C$

Also, $f(2) = 0$

$\therefore f(2) = (2)^4 + \frac{1}{(2)^3} + C = 0$

$\Rightarrow 16 + \frac{1}{8} + C = 0$

$\Rightarrow C = \frac{-129}{8}$

$\therefore f(x) = x^4 + \frac{1}{x^3} - \frac{129}{8}$

$\therefore$ Correct answer is A.

Exercise 7.2

1.

Integrate $\dfrac{2x}{1+x^2}$

Ans - Let $1 + x^2 = t$

$\therefore 2x\,dx = dt$

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{1}{t}\,dt$$

$= \log|t| + C$

$= \log[1 + x^2] + C$

$= \log(1 + x^2) + C$

where C is an arbitrary constant

2.

Integrate $\dfrac{(\log x)^2}{x}$

Ans - Let $\log x = t$

$\therefore \dfrac{1}{x}\,dx = dt$

$$\Rightarrow \int \frac{(\log|x|)^2}{x}\,dx = \int t^2\,dt$$

$= \dfrac{t^3}{3} + C$

$= \dfrac{(\log|x|)^3}{3} + C$

where C is an arbitrary constant

3.

Integrate $\dfrac{1}{x+x\log x}$

Ans - Given function can be written as

$$\frac{1}{x + x\log x} = \frac{1}{x(1 + \log x)}$$

$$\therefore \frac{1}{x} dx = dt$$

$$\Rightarrow \int \frac{1}{x(1 + \log x)} dx = \int_t^1 dt$$

$$= \log|t| + C$$

$$= \log|1 + \log x| + C$$

Where C is an arbitrary constant.

4.

Integrate $\sin x \cdot \sin(\cos x)$

Ans - Let $\cos x = t$

$$\therefore -\sin x\, dx = t$$

$$\Rightarrow \int \sin x \cdot \sin(\cos x) dx = -\int \sin t\, dt$$

$$= -[-\cos t] + C$$

$$= \cos(\cos x) + C$$

Where C is an arbitrary constant.

5.

Integrate $\sin(ax + b)\cos(ax + b)$

Ans - Given function can be rewritten as

$$\sin(ax + b)\cos(ax + b) = \frac{2\sin(ax + b)\cos(ax + b)}{2}$$

$$= \frac{\sin2(ax + b)}{2}$$

let $2(ax + b) = t$

$\therefore 2adx = dt$

$$\Rightarrow \int \frac{\sin2(ax + b)}{2} dx = \frac{1}{2} \int \frac{\sin t \, dt}{2a}$$

$$= \frac{1}{4a} [-\cos t] + C$$

$$= \frac{-1}{4a} \cos2(ax + b) + C$$

Where C is an arbitrary constant.

6.

Integrate $\sqrt{ax + b}$

Ans - Let $ax + b = t$

$\Rightarrow adx = dt$

$\therefore dx = \dfrac{1}{a}dt$

$\Rightarrow \displaystyle\int (ax + b)^{\frac{1}{2}}\, dx = \dfrac{1}{a}\int t^{\frac{1}{2}}\, dt$

$= \dfrac{1}{a}\left(\dfrac{t^{\frac{1}{2}}}{\dfrac{3}{2}}\right) + C$

$= \dfrac{2}{3a}(ax + b)^{\frac{3}{2}} + C$

Where C is an arbitrary constant.

7.

Integrate $x\sqrt{x + 2}$

Ans - Let $x + 2 = t$

$\therefore dx = dt$

$\Rightarrow \displaystyle\int x\sqrt{x + 2}\, dx = \int (t - 2)\sqrt{t}\, dt$

$= \displaystyle\int \dfrac{t^{\frac{5}{2}}}{\dfrac{5}{2}} - 2\left(\dfrac{t^{\frac{3}{2}}}{\dfrac{3}{2}}\right) + C$

$= \dfrac{2}{5}t^{\frac{5}{2}} - \dfrac{4}{3}t^{\frac{3}{2}} + C$

$= \dfrac{2}{5}(x + 2)^{\frac{5}{2}} - \dfrac{4}{3}(x + 2)^{\frac{3}{2}} + C$

Where C is an arbitrary constant.

8.

Integrate $x\sqrt{1 + 2x^2}$

Ans - Let $1 + 2x^2 = t$

$\therefore 4xdx = dt$

$\Rightarrow \int x\sqrt{1 + 2x^2}dx = \int \frac{\sqrt{t}}{4}dt$

$= \frac{1}{4}\int t^{\frac{1}{2}}dt$

$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$

$= \frac{1}{6}(1 + 2x^2)^{\frac{3}{2}} + C$

where C is an arbitrary constant

9.

Integrate $(4x + 2)\sqrt{x^2 + x + 1}$

Ans - Let $x^2 + x + 1 = t$

$\therefore (2x + 1)dx = dt$

$\int (4x + 2)\sqrt{x^2 + x + 1}dx$

$= \int 2\sqrt{t}dt = 2\int \sqrt{t}dt$

$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$

$= \frac{4}{3}(x^2 + x + 1)^{\frac{3}{2}} + C$

where C is an arbitrary constant

10.

Integrate $\dfrac{1}{x-\sqrt{x}}$

Ans - Given function can be rewritten as:

$$\frac{1}{x-\sqrt{x}} = \frac{1}{\sqrt{x}(\sqrt{x}-1)}$$

$\text{Let}(\sqrt{x}-1) = t$

$\therefore \dfrac{1}{2\sqrt{x}}dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{x}(\sqrt{x}-1)}dx = \int \frac{2}{t}dt$$

$= 2\log|t| + C$

$= 2\log\left|\sqrt{x}-1\right| + C$

where C is an arbitrary constant.

11.

Integrate $\dfrac{x}{\sqrt{x+4}}, x > 0$

Ans - Let $x + 4 = t$

$\therefore dx = dt$

$$\int \frac{x}{\sqrt{x+4}} dx = \int \frac{(t-4)}{\sqrt{t}} dt$$

$$= \int \left(\sqrt{t} - \frac{4}{\sqrt{t}} \right) dt$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4 \left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right) + C$$

$$= \frac{2}{3} (t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3} t \cdot t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$$

$$= \frac{2}{3} t^{\frac{1}{2}} (t - 12) + C$$

$$= \frac{2}{3} (x+4)^{\frac{1}{2}} (x + 4 - 12) + C$$

$$= \frac{2}{3} \sqrt{x+4} (x - 8) + C$$

where C is an arbitrary constant

12.

Ans - Let $x^3 - 1 = t$

$\therefore 3x^2\, dx = dt$

$\Rightarrow \int (x^3 - 1)^{\frac{1}{3}} x^5 dx = \int (x^3 - 1)^{\frac{1}{3}} x^3 x^2\, dx$

$= \int t^{\frac{1}{3}} (t + 1) \frac{dt}{3} = \frac{1}{3} \int \left(t^{\frac{4}{3}} + t^{\frac{1}{3}} \right) dt$

$= \frac{1}{3} \left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}} + \frac{t^{\frac{4}{3}}}{\frac{4}{3}} \right] + C$

$= \frac{1}{3} \left[\frac{3}{7} t^{\frac{7}{3}} + \frac{3}{4} t^{\frac{4}{3}} \right] + C$

$= \frac{1}{7} (x^3 - 1)^{\frac{7}{3}} + \frac{1}{4} (x^3 - 1)^{\frac{4}{3}} + C$

where C is an arbitrary constant.

13.

Integrate $\dfrac{x^2}{(2+3x^3)^3}$

Ans – Let $2 + 3x^3 = t$

$\therefore 9x^2 dx = dt$

$\Rightarrow \int \frac{x^2}{(2 + 3x^3)^3} dx = \frac{1}{9} \int \frac{dt}{(t)^3}$

$= \frac{1}{9} \left[\frac{t^{-2}}{-2} \right] + C$

$= \frac{-1}{18} \left(\frac{1}{t^2} \right) + C$

$= \frac{-1}{18(2 + 3x^3)^2} + C$

where C is an arbitrary constant

14.

Integrate $\dfrac{1}{x(\log x)^m}$, $x > 0$

Ans - Let $\log x = t$

$\dfrac{1}{x} dx = dt$

$\Rightarrow \int \dfrac{1}{x(\log x)^m} dx = \int \dfrac{dt}{(t)^m}$

$= \dfrac{t^{-m+1}}{-m+1} + C$

$= \dfrac{(\log x)^{1-m}}{(1-m)} + C$

where C is an arbitrarv constant.

15.

Integrate $\dfrac{x}{9-4x^2}$

Ans - Let $9 - 4x^2 = t$

$\therefore -8x\,dx = dt$

$\Rightarrow \int \dfrac{x}{9-4x^2} dx = \dfrac{-1}{8} \int \dfrac{1}{t} dt$

$= \dfrac{-1}{8} \log|t| + C$

$= \dfrac{-1}{8} \log|9 - 4x^2| + C$

where C is an arbitrary constant

16.

Integrate e^{2x+3}

Ans - Let $2x + 3 = t$

$\therefore 2dx = dt$

$\Rightarrow \int e^{2x+3} dx = \dfrac{1}{2} \int e^t dt$

$= \dfrac{1}{2} (e^t) + C$

$= \dfrac{1}{2} (e^{2x+3}) + C$

where C is an arbitrary constant.

17.

Integrate $\dfrac{x}{e^{x^2}}$

Ans - Let $x^2 = t$

$\therefore 2x dx = dt$

$\Rightarrow \int \dfrac{x}{e^{x^2}} dx = \dfrac{1}{2} \int \dfrac{1}{e^t} dt$

$= \dfrac{1}{2} \int e^{-t} dt$

$= \dfrac{1}{2} \left(\dfrac{e^{-t}}{-1} \right) + C$

$= -\dfrac{1}{2} e^{-x^2} + C$

$= \dfrac{-1}{2e^{x^2}} + C$

where C is an arbitrary constant

18.

Integrate $\dfrac{e^{\tan^{-1}x}}{1+x^2}$

Ans – Let $\tan^{-1}x = t$

$\therefore \dfrac{1}{1+x^2}dx = dt$

$\Rightarrow \int \dfrac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^t dt$

$= e^t + C$

$= e^{\tan^{-1}x} + C$

where C is an arbitrary constant

19.

Integrate $\dfrac{e^{2x}-1}{e^{2x}+1}$

Ans - Dividing the given function's numerator and denominator by e^x, we obtain

$\dfrac{\dfrac{(e^{2x}-1)}{e^x}}{\dfrac{(e^{2x}+1)}{e^x}} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Let $e^x + e^{-x} = t$

$(e^x - e^{-x})dx = dt$

$\Rightarrow \int \dfrac{e^{2x}-1}{e^{2x}+1}dx = \int \dfrac{e^x - e^{-x}}{e^x + e^{-x}}dx$

$= \int \dfrac{dt}{t}$

$= \log|t| + C$

$= \log|e^x - e^{-x}| + C$

where C is an arbitrary constant.

20.

Integrate $\dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$

Ans - Let $e^{2x} + e^{-2x} = t$

$\Rightarrow 2e^{2x} - 2e^{-2x}dx = dt$

$\Rightarrow 2(e^{2x} - e^{-2x})dx = dt$

$\Rightarrow \int \left(\dfrac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right)dx = \int \dfrac{dt}{2t}$

$= \dfrac{1}{2}\int \dfrac{1}{t}d$

$= \dfrac{1}{2}\log|t| + C$

$= \dfrac{1}{2}\log|e^{2x} + e^{-2x}| + C$

where C is an arbitrary constant.

21.

Integrate $\tan^2(2x + 3)$

Ans – $\tan^2(2x - 3) = \sec^2(2x - 3) - 1$

Let $2x - 3 = t$

$\therefore 2dx = dt$

$\Rightarrow \int \tan^2(2x - 3)dx = \int [\sec^2(2x - 3) - 1]dx$

$= \dfrac{1}{2} \int (\sec^2 t)dt - \int 1dx$

$= \dfrac{1}{2} \int \sec^2 t\, dt - \int 1dx$

$= \dfrac{1}{2} \tan t - x + C$

$= \dfrac{1}{2} \tan(2x - 3) - x + C$

where C is an arbitrary constant

22.

Integrate $\sec^2(7 - 4x)$

Ans – Let $7 - 4x = t$

$\therefore -4\, dx = dt$

$\therefore \int \sec^2(7 - 4x)dx = \dfrac{-1}{4} \int \sec^2 t\, dt$

$= \dfrac{-1}{4} (\tan t) + C$

$= \dfrac{-1}{4} \tan(7 - 4x) + C$

where C is an arbitrary constant

23.

Integrate $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

Ans - Let $\sin^{-1}x = t$

$$\frac{1}{\sqrt{1-x^2}}\,dx = dt$$

$$\Rightarrow \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int t\,dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{(\sin^{-1}x)^2}{2} + C$$

where C is an arbitrary constant.

24.

Integrate $\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

Ans - Given function is

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2(3\cos x + 2\sin x)}$$

Let $3\cos x + 2\sin x = t$

$(-3\sin x + 2\cos x)dx = dt$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2}\int \frac{1}{t}dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|2\sin x + 3\cos x| + C$$

Where C is an arbitrary constant

25.

Integrate $\dfrac{1}{\cos^2 x(1-\tan x)^2}$

Ans - Given function is,

$$\frac{1}{\cos^2 x(1-\tan x)^2} = \frac{\sec^2}{(1-\tan x)^2}$$

Let $(1 - \tan x) = t$

$-\sec^2 x \, dx = dt$

$$\Rightarrow \int \frac{\sec^2}{(1 - \tan x)^2} \, dx = \int \frac{-dt}{t^2}$$

$$= -\int t^{-2} \, dt$$

$$= +\frac{1}{t} + C$$

$$= \frac{1}{(1 - \tan x)} + C$$

where C is an arbitrary constant

26.

Integrate $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

Ans - Let $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}} \, dx = dt$$

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}} \, dx = 2 \int \cos t \, dt \ = 2 \cos t \, dt \ = \ 2\sin t + C$$

$$= 2\sin\sqrt{x} + C$$

where C is an arbitrary constant.

27.

Integrate $\sqrt{\sin 2x \cos 2x}$

Ans - Let $\sin 2x = t$

so, $2\cos 2x\,dx = dt$

$\Rightarrow \int \sqrt{\sin 2x}\cos 2x\,dx$

$= \dfrac{1}{2}\left(\dfrac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$

$= \dfrac{1}{2}\int \sqrt{t}\,dt$

$= \dfrac{1}{3}t^{\frac{3}{2}} + C$

$= \dfrac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$

where C is an arbitrary constant

28.

Integrate $\dfrac{\cos x}{\sqrt{1+\sin x}}$

Ans – Let $1 + \sin x = t$

$\therefore \cos x\,dx = dt$

$\Rightarrow \int \dfrac{\cos x}{\sqrt{1 + \sin x}}\,dx = \int \dfrac{dt}{\sqrt{t}}$

$= \dfrac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= 2\sqrt{t} + C$

$= 2\sqrt{1 + \sin x} + C$

where C is an arbitrary constant.

29.

Integrate cot x log sin x

Ans – Let log sinx = t

$$\Rightarrow \frac{1}{\sin x} \cdot \cos x dx = dt$$

$$\therefore \cot x dx = dt$$

$$\Rightarrow \int \cot x \log \sin x \, dx = \int t dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{1}{2} (\log \sin x)^2 + C$$

where C is an arbitrary constant.

30.

Integrate $\frac{\sin x}{1+\cos x}$

Ans – $\int \frac{\sin x}{1 + \cos x} dx = \int -\frac{dt}{t}$

$$= -\log|t| + C$$

$$= -\log|1 + \cos x| + C$$

where C is an arbitrary constant.

31.

Integrate $\frac{\sin x}{(1+\cos x)^2}$

Ans - Let $1 + \cos x = t$

$-\sin x\, dx = dt$

$$\Rightarrow \int \frac{\sin x}{(1 + \cos x)^2} dx = \int -\frac{dt}{t^2}$$

$$= -\int t^{-2} dt$$

$$= \frac{1}{t} + C$$

$$= \frac{1}{1 + \cos x} + C$$

where C is an arbitrary constant

32.

Integrate $\dfrac{1}{1+\cot x}$

Ans - Let $I = \int \frac{1}{1+\cot x} dx$

$= \int \frac{1}{1 + \frac{\cos x}{\sin x}} dx$

$= \int \frac{\sin x}{\sin x + \cos x} dx$

$= \frac{1}{2} \int \frac{2\sin x}{\sin x + \cos x} dx$

$= \frac{1}{2} \int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx$

$= \frac{1}{2} \int 1 dx + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

$= \frac{1}{2}(x) + \frac{1}{2} \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Let $\sin x + \cos x = t$

$\Rightarrow (\cos x - \sin x) dx = dt$

$\therefore I = \frac{x}{2} + \frac{1}{2} \int \frac{-(dt)}{t}$

$= \frac{x}{2} - \frac{1}{2} \log|t| + C$

$= \frac{x}{2} - \frac{1}{2} \log|\sin x + \cos x| + C$

where C is an arbitrary constant.

33.

Integrate $\frac{1}{1-\tan x}$

Ans – Let $I = \int \dfrac{1}{1-\tan x}\,dx$

$= \int \dfrac{1}{1 - \dfrac{\sin x}{\cos x}}\,dx$

$= \int \dfrac{\cos x}{\cos x - \sin x}\,dx$

$= \dfrac{1}{2}\int \dfrac{2\cos x}{\cos x - \sin x}\,dx$

$= \dfrac{1}{2}\int \dfrac{(\cos x - \sin x) + (\cos x + \sin x)}{(\cos x - \sin x)}\,dx$

$= \dfrac{1}{2}\int 1\,dx + \dfrac{1}{2}\int \dfrac{\cos x + \sin x}{\cos x - \sin x}\,dx$

$= \dfrac{x}{2} + \dfrac{1}{2}\int \dfrac{\cos x + \sin x}{\cos x - \sin x}\,dx$

Put $\cos x - \sin x = t \Rightarrow (-\sin x - \cos x)\,dx = dt$

$\therefore I = \dfrac{x}{2} + \dfrac{1}{2}\int \dfrac{-(dt)}{t}$

$= \dfrac{x}{2} - \dfrac{1}{2}\log|t| + C$

$= \dfrac{x}{2} - \dfrac{1}{2}\log|\cos x - \sin x| + C$

where C is an arbitrary constant.

34.

Integrate $\dfrac{\sqrt{\tan x}}{\sin x \cos x}$

Ans - Let $I = \int \dfrac{\sqrt{\tan x}}{\sin x \cos x} dx$

$= \int \dfrac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$

$= \int \dfrac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$

$= \int \dfrac{\sec^2 x \, dx}{\sqrt{\tan x}}$

Let $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$\therefore I = \int \dfrac{dt}{\sqrt{t}}$

$= 2\sqrt{t} + C$

$= 2\sqrt{\tan x} + C$

where C is an arbitrary constant

35.

Integrate $\dfrac{(1 + \log x)^2}{x}$

Ans – Let $1 + \log x = t$

$\therefore \dfrac{1}{x} dx = dt$

$\Rightarrow \int \dfrac{(1 + \log x)^2}{x} dx = \int t^2 dt$

$= \dfrac{t^3}{3} + C$

$= \dfrac{(1 + \log x)^3}{3} + C$

where C is an arbitrary constant

36.

Integrate $\dfrac{(x+1)(x+\log x)^2}{x}$

Ans - Given function can be rewritten as

$$\frac{(x+1)(x+\log x)^2}{x}$$

$$= \left(1 + \frac{1}{x}\right)(x + \log x)^2$$

Let $(x + \log x) = t$

$$\therefore \left(1 + \frac{1}{x}\right)dx = dt$$

$$\Rightarrow \int \left(1 + \frac{1}{x}\right)(x + \log x)^2 dx$$

$$= \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{1}{3}(x + \log x)^3 + C$$

Where C is an arbitrary constant.

37.

Integrate $\dfrac{x^3 \sin(\tan^{-1} x^4)}{1 + x^8}$

Ans - ∴ $4x^3 dx = dt$

$$\Rightarrow \int \frac{x^3 \sin(\tan^{-1}x^4)}{1+x^8} dx = \frac{1}{4} \int \frac{\sin(\tan^{-1}t)}{1+t^2} dt$$

Let $\tan^{-1}t = u$

$$\therefore \frac{1}{1+t^2} dt = du$$

From (1), we obtain

$$\int \frac{x^3 \sin(\tan^{-1}x^4)dx}{1+x^8} = \frac{1}{4} \int \sin u \, du$$

$$= \frac{1}{4}(-\cos u) + C$$

$$= -\frac{1}{4} \cos(\tan^{-1}t) + C$$

$$= \frac{-1}{4} \cos(\tan^{-1}x^4) + C$$

Where C is an arbitrary constant.

38.

Here, $\int \dfrac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$ equals

A. $10^x - x^{10} + C$

C. $(10^x - x^{10})^{-1} + C$

B. $10^x + x^{10} + C$

D. $\log(10^x + x^{10}) + C$

Ans - Let $x^{10} + 10^x = t$

$$\therefore (10x^9 + 10^x \log_e 10)dx = \int \frac{dt}{t}$$

$$\therefore \int \frac{10x^9 + 10^x \log_e 10}{10^x + 10x} dx = \int \frac{dt}{t}$$

$$= \log t + C$$

$$= \log(10^x + x^{10}) + C$$

Hence, correct answer is D.

39.

Here, $\int \frac{dx}{\sin^2 x \cos^2 x}$ **equals**

A. $\tan x + \cot x + C$

C. $\tan x \cot x + C$

B. $\tan x - \cot x + C$

D. $\tan x - \cot 2x + C$

Ans - Let $I = \int \frac{dx}{\sin^2 x \cos^2 x}$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \sec^2 x\, dx + \int \text{cosec}^2 x\, dx$$

$$= \tan x = \cot x + C$$

Hence, correct answer is B.

Exercise 7.3

1.

Find the integral of the function: sin²(2x + 5)

Ans - Given expression can be written as

$$\sin^2(2x + 5) = \frac{1 - \cos 2(2x + 5)}{2}$$

$$\Rightarrow \sin^2(2x + 5) = \frac{1 - \cos(4x + 10)}{2}$$

Integration of given expression is

$$\Rightarrow \int \sin^2(2x + 5)dx = \int \frac{1 - \cos(4x + 10)}{2} dx$$

$$\Rightarrow \int \sin^2(2x + 5)dx = \frac{1}{2}\int 1dx - \frac{1}{2}\int \cos(4x + 10)dx$$

$$\Rightarrow \int \sin^2(2x + 5)dx = \frac{1}{2}x - \frac{1}{2}\left(\frac{\sin(4x + 10)}{4}\right) + C$$

$$\therefore \int \sin^2(2x + 5)dx = \frac{1}{2}x - \frac{1}{8}\sin(4x + 10) + C$$

2.

Find the integral of the function: sin 3x cos 4x

Ans - Given expression sin 3x cos 4x

We know identity $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

Using this identity we can write given expression as

$\sin 3x \cos 4x = \frac{1}{2}\{\sin(3x+4x) + \sin(3x-4x)\}$

We can integrate above expression as

$\Rightarrow \int \sin 3x \cos 4x \, dx = \int \frac{1}{2}\{\sin 7x + \sin(-x)\}dx$

$\Rightarrow \int \sin 3x \cos 4x \, dx = \frac{1}{2}\int \{\sin 7x + \sin x\}dx$

$\Rightarrow \int \sin 3x \cos 4x \, dx = \frac{1}{2}\int \sin 7x \, dx + \frac{1}{2}\int \sin x \, dx$

$\Rightarrow \int \sin 3x \cos 4x \, dx = \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}(-\cos x) + C$

$\therefore \int \sin 3x \cos 4x \, dx = \frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$

3.

Find the integral of the function: cos 2x cos 4x cos 6x.

Ans - Given expression $\cos 2x \cos 4x \cos 6x$.

We know identity $\cos A \cos B = \frac{1}{2}\{\cos(A + B) + \cos(A - B)\}$

Using this identity we can write given expression as

$\cos 2x(\cos 4x \cos 6x) = \cos 2x \frac{1}{2}\{\cos(4x + 6x) + \cos(4x - 6x)\}$

Integration of the above expression is

$\Rightarrow \int \cos 2x(\cos 4x \cos 6x)$

$$= \int \cos 2x \left[\frac{1}{2}\{\cos 10x + \cos(-2x)\}\right] dx$$

$\Rightarrow \int \cos 2x(\cos 4x \cos 6x)$

$$= \int \left[\frac{1}{2}\{\cos 2x \cos 10x + \cos 2x \cos(-2x)\}\right] dx$$

$\Rightarrow \int \cos 2x(\cos 4x \cos 6x)$

$$= \frac{1}{2}\int \left[\{\cos 2x \cos 10x + \cos^2 2x\}\right] dx$$

Again, applying the identity

$\cos A \cos B = \frac{1}{2}\{\cos(A + B) + \cos(A - B)\}$, we get

$\Rightarrow \int \cos 2x(\cos 4x \cos 6x)$

$$= \frac{1}{2}\int \left[\left\{\frac{1}{2}\cos(2x + 10x) + \cos(2x - 10x) + \left(\frac{1 + \cos 4x}{2}\right)\right\}\right] dx$$

$\Rightarrow \int \cos 2x(\cos 4x \cos 6x) = \frac{1}{4}\int [\cos 12x + \cos 8x + \cos 4x] dx$

$\therefore \int \cos 2x \cos 4x \cos 6x = \frac{1}{4}\left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + C$

4.

Find the integral of the function: $\sin^3(2x + 1)$

Ans - Given expression $\sin^3(2x + 1)$

Let $I = \int \sin^3(2x + 1)dx$

$\Rightarrow I = \int \sin^2(2x + 1)\sin(2x + 1)dx$

$\Rightarrow I = \int (1 - \cos^2(2x + 1))\sin(2x + 1)dx$

Let $\cos(2x + 1) = t$

$\therefore -2\sin(2x + 1)dx = dt$

Integration becomes

$\Rightarrow I = -\dfrac{1}{2}\int (1 - t^2)dt$

$\Rightarrow I = -\dfrac{1}{2}\left(t - \dfrac{t^3}{3}\right) + C$

Substitute $\cos(2x + 1) = t$,

$\Rightarrow I = -\dfrac{1}{2}\left(\cos(2x + 1) - \dfrac{\cos^3(2x + 1)}{3}\right) + C$

$\therefore \int \sin^3(2x + 1)dx = \dfrac{-\cos(2x + 1)}{2} + \dfrac{\cos^3(2x + 1)}{3} + C$

5.

Find the integral of the function: $\sin^3 x \cos^3 x$

Ans - Given expression $\sin^3 x \cos^3 x$

Let $I = \int \sin^3 x \cos^3 x \, dx$

$\Rightarrow I = \int \sin^2 x \sin x \cos^3 x \, dx$

$\Rightarrow I = \int \cos^3 x (1 - \cos^2 x) \sin x \, dx$

Let $\cos x = t$

$\therefore - \sin x \, dx = dt$

Integration becomes

$\Rightarrow I = -\int t^3 - t^2) dt$

$\Rightarrow I = -\int (t^3 - t^5) dt$

$\Rightarrow I = [\frac{t^4}{4} - \frac{t^6}{6}] + C$

Substitute $\cos x = t$

$I = \left[\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6} \right] + C$

$\therefore \int \sin^3 x \cos^3 x \, dx = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$

6.

Find the integral of the function: $\sin x \, \sin 2x \, \sin 3x$.

Ans - Given expression sin x sin2x sin3x

Using the identity $\sin A \sin B = \frac{1}{2}\cos(A - B) - \cos(A + B)$, given expression can be written as

$$\sin x \sin 2x \sin 3x = \sin x.\frac{1}{2}\cos(2x - 3x) - \cos(2x + 3x)$$

Integration of given expression is

$$\Rightarrow \int \sin x \sin 2x \sin 3x \, dx$$
$$= \frac{1}{2}\int (\sin x \cos(-x) - \sin x \cos(5x))dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x \, dx = \frac{1}{2}\int (\sin x \cos x - \sin x \cos 5x)dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x \, dx$$
$$= \frac{1}{2}\int \frac{\sin 2x}{2} dx - \frac{1}{2}\int (\sin x \cos 5x)dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x \, dx$$
$$= \frac{1}{4}\left(\frac{-\cos 2x}{2}\right)$$
$$- \frac{1}{2}\int \left\{\frac{1}{2}(\sin(x + 5x) + \sin(x - 5x))\right\}dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x \, dx$$
$$= \frac{-\cos 2x}{8} - \frac{1}{4}\int \{(\sin 6x + \sin(-4x))\}dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x \, dx$$
$$= \frac{-\cos 2x}{8} - \frac{1}{4}\int \{(\sin 6x + \sin 4x)\}dx$$

$$\Rightarrow \int \sin x \sin 2x \sin 3x \, dx$$
$$= \frac{-\cos 2x}{8} - \frac{1}{8}\left|\frac{-\cos 6x}{3} + \frac{\cos 4x}{4}\right| + C$$

$$\therefore \int \sin x \sin 2x \sin 3x \, dx = \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{4} - \cos 2x\right] + C$$

7.

Find the integral of the function: sin 4x sin 8x.

Ans - Given expression sin 4x sin 8x

Using the identity $\sin A \sin B = \frac{1}{2}\cos(A - B) - \cos(A + B)$
given expression can be written as

$$\Rightarrow \sin 4x \sin 8x = \frac{1}{2}\cos(4x - 8x) - \cos(4x + 8x)$$

Integration of given expression is

$$\Rightarrow \int \sin4x \sin8x \, dx = \frac{1}{2}\int \left(\cos(-4x) - \cos(12x)\right)dx$$

$$\Rightarrow \int \sin4x \sin8x \, dx = \frac{1}{2}\int \left(\cos4x - \cos12x\right)dx$$

$$\therefore \int \sin4x \sin8x \, dx = \frac{1}{2}\left[\frac{\sin4x}{4} - \frac{\sin12x}{12}\right] + C$$

8.

Find the integral of the function: $\frac{1-\cos x}{1+\cos x}$

Ans - Given expression $\dfrac{1-\cos x}{1+\cos x}$

Using identities $2\sin^2\dfrac{x}{2} = 1 - \cos x$ and $\cos x = 2\cos^2\dfrac{x}{2} - 1$,

Given expression can be writen as,

$\Rightarrow \dfrac{1-\cos x}{1+\cos x} = \dfrac{2\sin^2\dfrac{x}{2}}{2\cos^2\dfrac{x}{2}}$

$\Rightarrow \dfrac{1-\cos x}{1+\cos x} = \tan^2\dfrac{x}{2}$

Integration of given expression is

$\Rightarrow \displaystyle\int \dfrac{1-\cos x}{1+\cos x}\,dx = \int [\tan^2\dfrac{x}{2}]dx$

$\Rightarrow \displaystyle\int \dfrac{1-\cos x}{1+\cos x}\,dx = \int [\sec^2\dfrac{x}{2} - 1]dx$

$\Rightarrow \displaystyle\int \dfrac{1-\cos x}{1+\cos x}\,dx = \left[\dfrac{\tan\dfrac{x}{2}}{\dfrac{1}{2}} - x\right] + C$

$\therefore \displaystyle\int \dfrac{1-\cos x}{1+\cos x}\,dx = 2\tan\dfrac{x}{2} - x + C$

9.

Ans - Given expression $\dfrac{\cos x}{1 + \cos x}$

Using identities $2\sin^2 \dfrac{x}{2} = 1 - \cos x$ and $\cos x = 2\cos^2 \dfrac{x}{2} - 1$,

Given expression can be writen as

$$\Rightarrow \frac{\cos x}{1 + \cos x} = \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}$$

$$\Rightarrow \frac{\cos x}{1 + \cos x} = \frac{1}{2}\left[1 - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}\right]$$

$$\Rightarrow \frac{\cos x}{1 + \cos x} = \frac{1}{2}\left[1 - \tan^2 \frac{x}{2}\right]$$

Integration of given expression is

$$\Rightarrow \int \frac{\cos x}{1 + \cos x}\,dx = \frac{1}{2}\int \left[1 - \tan^2 \frac{x}{2}\right] dx$$

$$\Rightarrow \int \frac{\cos x}{1 + \cos x}\,dx = \frac{1}{2}\int \left[1 - \sec^2 \frac{x}{2} + 1\right]dx$$

$$\Rightarrow \int \frac{\cos x}{1 + \cos x}\,dx = \frac{1}{2}\int \left[2 - \sec^2 \frac{x}{2}\right]dx$$

$$\Rightarrow \int \frac{\cos x}{1 + \cos x}\,dx = \frac{1}{2}\left[2x - \frac{\tan \frac{x}{2}}{\frac{1}{2}}\right] + C$$

$$\therefore \int \frac{\cos x}{1 + \cos x}\,dx = x - \tan \frac{x}{2} + C$$

10.

Find the integral of the function: $\sin^4 x$

Ans - Given expression can be written as $\sin^4 x = \sin^2 x \sin^2 x$

$$\Rightarrow \sin^4 x = \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 - \cos 2x}{2}\right)$$

$$\Rightarrow \sin^4 x = \frac{1}{4}(1 - \cos 2x)^2$$

$$\Rightarrow \sin^4 x = \frac{1}{4}(1 + \cos^2 2x - 2\cos 2x)$$

$$\Rightarrow \sin^4 x = \frac{1}{4}\left(1 + \frac{1 + \cos 4x}{2} - 2\cos 2x\right)$$

$$\Rightarrow \sin^4 x = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{\cos 4x}{2} - 2\cos 2x\right)$$

$$\Rightarrow \sin^4 x = \frac{1}{4}\left(\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right)$$

Integration of given expression is

$$\Rightarrow \int \sin^4 x\, dx = \frac{1}{4}\int \left(\frac{3}{2} + \frac{1}{2}\cos 4x - 2\cos 2x\right) dx$$

$$\Rightarrow \int \sin^4 x\, dx = \frac{1}{4}\left[\frac{3}{2}x + \frac{\sin 4x}{8} - \sin 2x\right] + C$$

$$\therefore \int \sin^4 x\, dx = \frac{3x}{8} + \frac{\sin 4x}{32} - \frac{1}{4}\sin 2x + C$$

11.

Find the integral of the function: $\cos^4 2x$.

Ans – Given expression can be written as

$$\cos^4 2x = (\cos^2 2x)^2$$

$$\Rightarrow \cos^4 2x = \left(\frac{1 + \cos 4x}{2}\right)^2$$

$$\Rightarrow \cos^4 2x = \frac{1}{4}(1 + \cos^2 4x + 2\cos 4x)$$

$$\Rightarrow \cos^4 2x = \frac{1}{4}\left(1 + \frac{1 + \cos 8x}{2} + 2\cos 4x\right)$$

$$\Rightarrow \cos^4 2x = \frac{1}{4}\left(1 + \frac{1}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right)$$

$$\Rightarrow \cos^4 2x = \frac{1}{4}\left(\frac{3}{2} + \frac{\cos 8x}{2} + 2\cos 4x\right)$$

Integration of given expression is

$$\Rightarrow \int \cos^4 2x\, dx = \int \left(\frac{3}{8} + \frac{\cos 8x}{8} + \frac{\cos 4x}{2}\right) dx$$

$$\therefore \int \cos^4 2x\, dx = \frac{3}{8}x + \frac{\sin 8x}{64} + \frac{\sin 4x}{8} + C$$

12.

Find the integral of the function: $\dfrac{\sin^2 x}{1 + \cos x}$

Ans - Given expression $\dfrac{\sin^2 x}{1+\cos x}$

Using identity $\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2}$ and $\cos x = 2\cos^2\dfrac{x}{2} - 1$,

Given expression can be written as

$$\Rightarrow \dfrac{\sin^2 x}{1+\cos x} = \dfrac{\left(2\sin\dfrac{x}{2}\cos\dfrac{x}{2}\right)^2}{2\cos^2\dfrac{x}{2}}$$

$$\Rightarrow \dfrac{\sin^2 x}{1+\cos x} = \dfrac{4\sin^2\dfrac{x}{2}\cos^2\dfrac{x}{2}}{2\cos^2\dfrac{x}{2}}$$

$$\Rightarrow \dfrac{\sin^2 x}{1+\cos x} = 2\sin^2\dfrac{x}{2}$$

$$\Rightarrow \dfrac{\sin^2 x}{1+\cos x} = 1 - \cos x$$

Integration of given expression is

$$\Rightarrow \int \dfrac{\sin^2 x}{1+\cos x}\,dx = \int 1\,dx - \int \cos x\,dx$$

$$\therefore \int \dfrac{\sin^2 x}{1+\cos x}\,dx = x - \sin x + C$$

13.

Find the integral of the function: $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

Ans - Given expression $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$

Using identity $\cos C - \cos D = -2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$, we get

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{-2\sin\dfrac{2x+2\alpha}{2}\sin\dfrac{2x-2\alpha}{2}}{-2\sin\dfrac{x+\alpha}{2}\sin\dfrac{x-\alpha}{2}}$$

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{\sin\dfrac{2(x+\alpha)}{2}\sin\dfrac{2(x-\alpha)}{2}}{\sin\dfrac{x+\alpha}{2}\sin\dfrac{x-\alpha}{2}}$$

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\dfrac{x+\alpha}{2}\sin\dfrac{x-\alpha}{2}}$$

Using identity $\sin 2x = 2\sin x \cos x$, we get

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$$
$$= \frac{\left[2\sin\dfrac{2(x+\alpha)}{2}\sin\dfrac{2(x+\alpha)}{2}\right]\left[2\sin\dfrac{2(x-\alpha)}{2}\sin\dfrac{2(x-\alpha)}{2}\right]}{\sin\dfrac{x+\alpha}{2}\sin\dfrac{x-\alpha}{2}}$$

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} = 4\cos\frac{x+\alpha}{2}\cos\frac{x-\alpha}{2}$$

$$\Rightarrow \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$$
$$= 2\left[\cos\frac{x+\alpha}{2}+\frac{x-\alpha}{2}+\cos\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right]$$

Integration of given expression is

$$\Rightarrow \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\,dx = 2\int [\cos x + \cos \alpha]\,dx$$

$$\therefore \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\,dx = 2[\sin x + x\cos \alpha] + C$$

14.

Find the integral of the function: $\frac{\cos x - \sin x}{1 + \sin 2x}$

Ans - Given expression $\frac{\cos x - \sin x}{1 + \sin 2x}$

We know that $\sin^2 x + \cos^2 x = 1$

Given expression can be written as

$$\Rightarrow \frac{\cos x - \sin x}{1 + \sin 2x} = \frac{\cos x - \sin x}{\sin^2 x + \cos^2 x + \sin 2x}$$

We can apply the identity $\sin 2x = 2\sin x \cos x$, we get

$$\Rightarrow \frac{\cos x - \sin x}{1 + \sin 2x} = \frac{\cos x - \sin x}{\sin^2 x + \cos^2 x + 2\sin x \cos x}$$

$$\Rightarrow \frac{\cos x - \sin x}{1 + \sin 2x} = \frac{\cos x - \sin x}{(\sin x + \cos x)^2}$$

Let $\sin x + \cos x = t$

$\therefore (\cos x - \sin x)dx = dt$

Integration of given expression is

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = \int \frac{dt}{t^2}$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = \int t^{-2} dt$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = -t^{-1} + C$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = -\frac{1}{t} + C$$

Substitute $\sin x + \cos x = t$,

$$\therefore \int \frac{\cos x - \sin x}{1 + \sin 2x} dx = -\frac{1}{\sin x + \cos x} + C$$

15.

Find the integral of the function: $\tan^3 2x \sec 2x$

Ans - Given expression $\tan^3 2x \sec 2x$

Given expression can be written as

$\Rightarrow \tan^3 2x \sec 2x = \tan^2 2x \tan 2x \sec 2x$

$\Rightarrow \tan^3 2x \sec 2x = (\sec^2 2x - 1) \tan 2x \sec 2x$

$\Rightarrow \tan^3 2x \sec 2x = \sec^2 2x \tan 2x \sec 2x - \tan 2x \sec 2x$

Integration of given expression is

$\Rightarrow \int \tan^3 2x \sec 2x \, dx$

$\qquad = \int \sec^2 2x \tan 2x \sec 2x \, dx - \int \tan 2x \sec 2x \, dx$

$\Rightarrow \int \tan^3 2x \sec 2x \, dx = \int \sec^2 2x \tan 2x \sec 2x \, dx - \dfrac{\sec 2x}{2} + C$

Let $\sec 2x = t$

$\therefore 2 \sec 2x \tan 2x \, dx = dt$

$\Rightarrow \int \tan^3 2x \sec 2x \, dx = \dfrac{1}{2} \int t^2 dt - \dfrac{\sec 2x}{2} + C$

$\Rightarrow \int \tan^3 2x \sec 2x \, dx = \dfrac{t^3}{6} - \dfrac{\sec 2x}{2} + C$

Substitute $\sec 2x = t$,

$\therefore \int \tan^3 2x \sec 2x \, dx = \dfrac{(\sec 2x)^3}{6} - \dfrac{\sec 2x}{2} + C$

16.

Find the integral of the function: $\tan^4 x$

Ans - Given expression can be written as

$\tan^4 x = \tan^2 x \tan^2 x$

$\Rightarrow \tan^4 x = (\sec^2 x - 1)\tan^2 x$

$\Rightarrow \tan^4 x = \sec^2 x \tan^2 x - \tan^2 x$

$\Rightarrow \tan^4 x = \sec^2 x \tan^2 x - (\sec^2 x - 1)$

$\Rightarrow \tan^4 x = \sec^2 x \tan^2 x - \sec^2 x + 1$

Integration of given expression is

$\Rightarrow \int \tan^4 x\, dx = \int (\sec^2 x \tan^2 x - \sec^2 x + 1)dx$

$\Rightarrow \int \tan^4 x\, dx = \int (\sec^2 x \tan^2 x)dx - \int \sec^2 x\, dx + \int 1\, dx$

$\Rightarrow \int \tan^4 x\, dx = \int \sec^2 x \tan^2 x\, dx - \tan x + x + C$

Let $\tan x = t$

$\therefore \sec^2 x\, dx = dt$

$\Rightarrow \int \tan^4 x\, dx = \int t^2 dt - \tan x + x + C$

$\Rightarrow \int \tan^4 x\, dx = \dfrac{t^3}{3} - \tan x + x + C$

Substitute $\tan x = t$,

$\therefore \int \tan^4 x\, dx = \dfrac{1}{3}\tan^3 x - \tan x + x + C$

17.

Find the integral of the function: $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Ans - Given expression can be written as

$$\Rightarrow \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}$$

$$\Rightarrow \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$

$$\Rightarrow \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \tan x \sec x + \cot x \, \mathrm{cosec}\, x$$

Integration of given expression is

$$\Rightarrow \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \int \tan x \sec x \, dx + \int \cot x \, \mathrm{cosec}\, x \, dx$$

$$\Rightarrow \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} \, dx = \sec x - \mathrm{cosec}\, x + C$$

18.

Find the integral of the function: $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

Ans - Given expression $\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

By applying the identity $\cos 2x = 1 - 2\sin^2 x$ we get

$$\Rightarrow \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} = \frac{\cos 2x + 1 - \cos 2x}{\cos^2 x}$$

$$\Rightarrow \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\Rightarrow \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} = \sec^2 x$$

Integration of given expression is

$$\Rightarrow \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \int \sec^2 x \, dx$$

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \tan x + C$$

19.

Find the integral of the function: $\frac{1}{\sin x \cos^3 x}$

Ans - Given expression $\dfrac{1}{\sin x \cos^3 x}$

We can apply the identity $\sin^2 x + \cos^2 x = 1$, we get

$\Rightarrow \dfrac{1}{\sin x \cos^3 x} = \dfrac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x}$

$\Rightarrow \dfrac{1}{\sin x \cos^3 x} = \dfrac{\sin^2 x}{\sin x \cos^3 x} + \dfrac{\cos^2 x}{\sin x \cos^3 x}$

$\Rightarrow \dfrac{1}{\sin x \cos^3 x} = \dfrac{\sin x}{\cos^3 x} + \dfrac{1}{\sin x \cos x}$

$\Rightarrow \dfrac{1}{\sin x \cos^3 x} = \tan x \sec^2 x + \dfrac{\dfrac{\cos^2 x}{\sin x \cos x}}{\cos^2 x}$

$\Rightarrow \dfrac{1}{\sin x \cos^3 x} = \tan x \sec^2 x + \dfrac{\sec^2 x}{\tan x}$

Integration of given expression is

$\Rightarrow \int \dfrac{1}{\sin x \cos^3 x} dx = \int \tan x \sec^2 x\, dx + \int \dfrac{\sec^2 x}{\tan x} dx$

Let $\tan x = t$

$\therefore \sec^2 x\, dx = dt$

$\Rightarrow \int \dfrac{1}{\sin x \cos^3 x} dx = \int t\, dt + \int \dfrac{1}{t} dt$

$\Rightarrow \int \dfrac{1}{\sin x \cos^3 x} dx = \dfrac{t^2}{2} + \log|t| + C$

Substitute $\tan x = t$,

$\therefore \int \dfrac{1}{\sin x \cos^3 x} dx = \dfrac{1}{2}\tan^2 x + \log|\tan x| + C$

20.

Find the integral of the function: $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$

Ans - Given expression can be written as

$$\frac{\cos 2x}{(\cos x + \sin x)^2} = \frac{\cos 2x}{\cos^2 x + \sin^2 x + 2\sin x \cos x}$$

We know that $\sin^2 x + \cos^2 x = 1$ and $2\sin x \cos x = \sin 2x$ we get

$$\Rightarrow \frac{\cos 2x}{(\cos x + \sin x)^2} = \frac{\cos 2x}{1 + \sin 2x}$$

Integration of given expression is

$$\Rightarrow \int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx = \int \frac{\cos 2x}{1 + \sin 2x}\, dx$$

Let $1 + \sin 2x + t$

$\therefore 2\cos 2x\, dx = dt$

Integration becomes

$$\Rightarrow \int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx = \frac{1}{2} \int \frac{1}{t}\, dt$$

$$\Rightarrow \int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx = \frac{1}{2} \log|t| + C$$

Substitute $1 + \sin 2x = t$

$$\Rightarrow \int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx = \frac{1}{2} \log|1 + \sin 2x| + C$$

$$\Rightarrow \int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx = \frac{1}{2} \log|(\cos x + \sin x)^2| + C$$

$$\therefore \int \frac{\cos 2x}{(\cos x + \sin x)^2}\, dx = \log|(\cos x + \sin x)| + C$$

21.

Find the integral of the function: $\sin^{-1}(\cos x)$.

Ans - Given expression $\sin^{-1}(\cos x)$.

Let $\cos x = t$

$\therefore \sin x = \sqrt{1 - t^2}$

$\Rightarrow -\sin x\, dx = dt$

$\Rightarrow dx = -\dfrac{dt}{\sin x}$

$\Rightarrow dx = -\dfrac{dt}{\sqrt{1 - t^2}}$

Integration of given expression is

$\Rightarrow \int \sin^{-1}(\cos x)dx = \int \sin^{-1} t \left(\dfrac{-dt}{\sqrt{1 - t^2}}\right)$

$\Rightarrow \int \sin^{-1}(\cos x)dx = -\int \left(\dfrac{\sin^{-1} t}{\sqrt{1 - t^2}}\right) dt$

Let $\sin^{-1} t = u$

$\Rightarrow \dfrac{1}{\sqrt{1 - t^2}} dt = du$

Integration becomes

$\Rightarrow \int \sin^{-1}(\cos x)dx = \int 4du$

$\Rightarrow \int \sin^{-1}(\cos x)dx = -\dfrac{u^2}{2} + C$

Substitute $\sin^{-1}t = u$

$\Rightarrow \int \sin^{-1}(\cos x)dx = -\dfrac{(\sin^{-1}t)^2}{2} + C$

Substitute $\cos x = t$

$\Rightarrow \int \sin^{-1}(\cos x)dx = -\dfrac{[\sin^{-1}(\cos x)]^2}{2} + C \dots\dots (1)$

We know that $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

$\therefore \sin^{-1}(\cos x) = \dfrac{\pi}{2} - \cos^{-1}(\cos x) = \left(\dfrac{\pi}{2} - x\right)$

Substitute in eq. (1), we get

22.

Find the integral of the function: $\dfrac{1}{\cos(x-a)\cos(x-b)}$

Ans - Given expression can be written as

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$

$$\Rightarrow \frac{1}{\cos(x-a)\cos(x-b)}$$
$$= \frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$$

$$\Rightarrow \frac{1}{\cos(x-a)\cos(x-b)}$$
$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right]$$

$$\Rightarrow \frac{1}{\cos(x-a)\cos(x-b)}$$
$$= \frac{1}{\sin(a-b)}[\tan(x-b)-\tan(x-b)]$$

Integration of given expression is

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$$
$$= \int \frac{1}{\sin(a-b)}[\tan(x-b)-\tan(x-b)]dx$$

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$$
$$= \int \frac{1}{\sin(a-b)}[-\log|\cos(x-b)| + \log\cos(x-a)]$$

$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$$
$$= \int \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right] + C$$

23.

Solve the following: $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to

A. $\tan x + \cot x + C$

B. $\tan x + \operatorname{cosec} x + C$

C. $-\tan x + \cot x + C$

D. $\tan x + \sec x + C$

Ans - Given expression $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$

Given expression can be written as

$$\Rightarrow \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx - \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$\Rightarrow \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \sec^2 x \, dx - \int \operatorname{cosec}^2 x \, dx$$

$$\therefore \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \tan x + \cot x + C$$

∴ Option A is correct answer

24.

Solve the following: $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$ equals

A. $-\cot(ex^x) + C$

B. $\tan(xe^x) + C$

C. $\tan(e^x) + C$

D. $\cot(e^x) + C$

Ans - Given expression $\int \frac{e^x(1+x)}{\cos^2(e^x x)} dx$

Let $e^x x = t$

$\therefore (e^x . x + e^x . 1)dx = dt$

$\Rightarrow e^x(x + 1)dx = dt$

Integration of given expression is

$\Rightarrow \int \frac{e^x(1 + x)}{\cos^2(e^x x)} dx = \int \frac{dt}{\cos^2 t}$

$\Rightarrow \int \frac{e^x(1 + x)}{\cos^2(e^x x)} dx = \int \sec^2 t \, dt$

$\Rightarrow \int \frac{e^x(1 + x)}{\cos^2(e^x x)} dx = \tan t + C$

Substitute $e^x x = t$,

$\therefore \int \frac{e^x(1 + x)}{\cos^2(e^x x)} dx = \tan(e^x x) + C$

$\therefore$ Option B is the correct answer

Exercise 7.4

1.

Integrate the function $\frac{3x^2}{x^6+1}$.

Ans - Let $x^3 = t$

Now, differentiate both sides

$3x^2 dx = dt$

$$\int \frac{3x^2}{x^6 + 1} dx = \int \frac{3x^2}{(x^3)^2 + 1}$$

$$\int \frac{3x^2}{x^6 + 1} dx = \int \frac{1}{t^2 + 1} dx$$

$$\int \frac{3x^2}{x^6 + 1} dx = \tan^{-1} t + C$$

$$\int \frac{3x^2}{x^6 + 1} dx = \tan^{-1}(x^3) + C$$

Where C is an arbitrary constant.

2.

Integrate the function $\dfrac{1}{\sqrt{1+4x^2}}$

Ans - Let $2x = t$

Now, differentiate both sides

$2dx = dt$

$dx = \dfrac{dt}{2}$

$\int \dfrac{1}{\sqrt{1+4x^2}} dx = \int \dfrac{1}{\sqrt{1+(2x)^2}} dx$

$\int \dfrac{1}{\sqrt{1+4x^2}} dx = \int \dfrac{1}{\sqrt{1+t^2}} \left(\dfrac{dt}{2}\right)$

$\int \dfrac{1}{\sqrt{1+4x^2}} dx = \dfrac{1}{2}\int \dfrac{1}{\sqrt{1+t^2}} dt$

$\int \dfrac{1}{\sqrt{1+4x^2}} dx = \dfrac{1}{2}\left(\log\left|t+\sqrt{1+t^2}\right|\right)$

$+C \left[\int \dfrac{1}{\sqrt{x^2+a^2}} dx = \log\left|x+\sqrt{x^2+a^2}\right|\right]$

$\int \dfrac{1}{\sqrt{1+4x^2}} dx = \dfrac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + C$

Where C is an arbitrary constant.

3.

Integrate the function $\dfrac{1}{\sqrt{(2-x)^2+1}}$

Ans - Let $2 - x = t$

Now, differentiate both sides

$0 - dx = dt$

$dx = -dt$

$$\int \frac{1}{\sqrt{(2-x)^2 + 1}} dx = \int \frac{1}{\sqrt{t^2 + 1}} (-dt)$$

$$\int \frac{1}{\sqrt{(2-x)^2 + 1}} dx = -\int \frac{1}{\sqrt{1 + t^2}} dt$$

$$\int \frac{1}{\sqrt{(2-x)^2 + 1}} dx = -\left(\log \left| t + \sqrt{1 + t^2} \right| \right)$$

$$+ C \left[\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log \left| x + \sqrt{x^2 + a^2} \right| \right]$$

$$\int \frac{1}{\sqrt{(2-x)^2 + 1}} dx = -\log \left| (2-x) + \sqrt{(2-x)^2 + 1} \right| + C$$

$$\int \frac{1}{\sqrt{(2-x)^2 + 1}} dx = \log \left(\frac{1}{(2-x) + \sqrt{x^2 - 4x + 5}} \right) + C$$

Where C is an arbitrary constant

4.

Integrate the function $\dfrac{1}{\sqrt{9 - 25x^2}}$

Ans - Let $5x = t$

$5dx = dt$

$dx = \dfrac{dt}{5}$

$$\int \frac{1}{\sqrt{9 - 25x^2}} dx = \int \frac{1}{\sqrt{(3)^2 - (5x)^2}} dx$$

$$\int \frac{1}{\sqrt{9 - 25x^2}} dx = \int \frac{1}{\sqrt{(3)^2 - (t)^2}} \left(\frac{dt}{5}\right)$$

$$\int \frac{1}{\sqrt{9 - 25x^2}} dx = \frac{1}{5} \int \frac{1}{\sqrt{(3)^2 - (t)^2}}$$

$$\int \frac{1}{\sqrt{9 - 25x^2}} dx = \frac{1}{5} \sin^{-1} \left(\frac{t}{3}\right) + C \left[\int \frac{1}{a^2 - x^2} dx = \sin^{-1} \frac{x}{a}\right]$$

$$\int \frac{1}{\sqrt{9 - 25x^2}} dx = \frac{1}{5} \sin^{-1} \left(\frac{5x}{3}\right) + C$$

Where C is an arbitrary constant.

5.

Integrate the function $\dfrac{3x}{1+2x^4}$.

Ans - Let $\sqrt{2}x^2 = t$

Now, differentiate both sides,

$2\sqrt{2}xdx = dt$

$xdx = \dfrac{dt}{2\sqrt{2}}$

$\displaystyle\int \dfrac{3x}{1+2x^4}dx = \int \dfrac{3x}{1+(2x^2)^2}dx$

$\displaystyle\int \dfrac{3x}{1+2x^4}dx = \int \dfrac{3}{1+t^2}\left(\dfrac{dt}{2\sqrt{2}}\right)$

$\displaystyle\int \dfrac{3x}{1+2x^4}dx = \dfrac{3}{2\sqrt{2}}\int \dfrac{3}{1+t^2}dt$

$\displaystyle\int \dfrac{3x}{1+2x^4}dx = \dfrac{3}{2\sqrt{2}}\tan^{-1}t + C\left[\int \dfrac{1}{1+x^2}dx = \tan^{-1}x + C\right]$

$\displaystyle\int \dfrac{3x}{1+2x^4}dx = \dfrac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x^2) + C$

Where C is an arbitrary constant.

6.

Integrate the function $\dfrac{x^2}{1-x^6}$

Ans - Let $x^3 = t$

Now, differentiate both sides

$3x^2 dx = dt$

$x^2 dx = \dfrac{dt}{3}$

$\displaystyle\int \dfrac{x^2}{1-x^6} dx = \int \dfrac{x^2}{1-(x^3)^2} dx$

$\displaystyle\int \dfrac{x^2}{1-x^6} dx = \int \dfrac{1}{1-t^2} \left(\dfrac{dt}{3}\right)$

$\displaystyle\int \dfrac{x^2}{1-x^6} dx = \dfrac{1}{3}\int \dfrac{1}{1-t^2} dt$

$\displaystyle\int \dfrac{x^2}{1-x^6} dx = \dfrac{1}{3}\left(\dfrac{1}{2}\log\left|\dfrac{1+t}{1-t}\right|\right) + C \left[\int \dfrac{1}{a^2-x^2} dx\right.$

$$= \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right| + C$$

$\displaystyle\int \dfrac{x^2}{1-x^6} dx = \dfrac{1}{3}\left(\dfrac{1}{2}\log\left|\dfrac{1+x^3}{1-x^3}\right|\right) + C$

Where C is an arbitrary constant.

7.

Integrate the function $\dfrac{x-1}{\sqrt{x^2-1}}$

Ans –

$$\int \frac{x}{\sqrt{x^2 - 1}} dx = \int \frac{x}{x^2 - 1} dx = \frac{1}{\sqrt{x^2 - 1}} dx \quad \dots \dots (1)$$

For $\int \frac{x}{\sqrt{x^2 - 1}} dx$

Now differentiate both sides,

$$2x dx = dt$$

$$x dx = \frac{dt}{2}$$

$$\int \frac{x}{\sqrt{x^2 - 1}} dx = \int \frac{1}{\sqrt{t}} \left(\frac{dt}{2} \right)$$

$$\int \frac{x}{\sqrt{x^2 - 1}} dx = \frac{1}{2} \int t^{-\frac{1}{2}} dt$$

$$\int \frac{x}{\sqrt{x^2 - 1}} dx = \frac{1}{2} \left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) + C$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \frac{1}{2}(2\sqrt{t}) + C$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \sqrt{t} + C$$

$$\int \frac{x}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} + C$$

From equation (1) we get,

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \int \frac{x}{\sqrt{x^2-1}} dx - \int \frac{1}{\sqrt{x^2-1}} dx$$

$$\int \frac{x-1}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} - \log\left|x + \sqrt{x^2-1}\right|$$

$$+ C \left[\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x + \sqrt{x^2-a^2}\right|\right]$$

Where C is an arbitrary constant.

8.

Integrate the function $\dfrac{x^2}{\sqrt{x^6+a^6}}$

Ans - Let $x^3 = t$

Now, differentiate both sides,

$3x^2 dx = dt$

$x^2 dx = \dfrac{dt}{3}$

$$\int \dfrac{x^2}{\sqrt{x^6 + a^6}} \, dx = \int \dfrac{x^2}{\sqrt{(x^3)^2 + (a^3)^2}} \, dx$$

$$\int \dfrac{x^2}{\sqrt{x^6 + a^6}} \, dx = \int \dfrac{1}{\sqrt{(t)^2 + (a^3)^2}} \left(\dfrac{dt}{3}\right)$$

$$\int \dfrac{x^2}{\sqrt{x^6 + a^6}} \, dx = \dfrac{1}{3} \int \dfrac{1}{\sqrt{(t)^2 + (a^3)^2}} \, dt$$

$$\int \dfrac{x^2}{\sqrt{x^6 + a^6}} \, dx = \dfrac{1}{3} \log \left| t + \sqrt{(t)^2 + (a^3)^2} \right|$$

$$+ C \left[\int \dfrac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| \right]$$

$$\int \dfrac{x^2}{\sqrt{x^6 + a^6}} \, dx = \dfrac{1}{3} \log \left| x + \sqrt{x^6 + a^6} \right| + C$$

Where C is an arbitrary constant.

9.

Integrate the function $\dfrac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$.

Ans - Let $\tan x = t$

Now, differentiate both sides

$\sec^2x\, dx = dt$

$$\int \frac{\sec^2x}{\sqrt{\tan^2x + 4}}\, dx = \int \frac{\sec^2x}{\sqrt{(\tan x)^2 + (2)^2}}\, dx$$

$$\int \frac{\sec^2x}{\sqrt{\tan^2x + 4}}\, dx = \int \frac{dt}{\sqrt{t^2 + (2)^2}}$$

$$\int \frac{\sec^2x}{\sqrt{\tan^2x + 4}}\, dx$$

$$= \log\left| t + \sqrt{t^2 + (2)^2} \right|$$

$$+ C \left[\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log\left| x + \sqrt{x^2 + a^2} \right| \right]$$

$$\int \frac{\sec^2x}{\sqrt{\tan^2x + 4}}\, dx = \log\left| \tan x + \sqrt{\tan^2x + 4} \right| + C$$

Where C is an arbitrary constant.

10.

Integrate the function $\dfrac{1}{\sqrt{x^2+2x+2}}$

Ans - Let $x + 1 = t$

Now, differentiate both sides

$dx = dt$

Now,

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x + 1)^2 + (1)^2}} dx$$

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(t)^2 + (1)^2}} dt$$

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx$$

$$= \log \left| t + \sqrt{t^2 + 1} \right|$$

$$+ C \left[\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log \left| x + \sqrt{x^2 + a^2} \right| \right]$$

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \log |(x + 1) + \sqrt{(x + 1)^2 + 1}| + C$$

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \log \left| x + 1 + \sqrt{x^2 + 2x + 2} \right| + C$$

Where C is an arbitrary constant.

11.

Find the integration $\frac{1}{\sqrt{9x^2 + 6x + 5}}$

Ans - Let $3x + 1 = t$

Now, differentiate both sides

$3dx = dt$

$dx = \dfrac{dt}{3}$

$$\int \frac{1}{\sqrt{9x^2 + 6x + 5}} dx = \frac{1}{\sqrt{9x^2 + 6x + 1 + 4}} dx$$

$$\int \frac{1}{\sqrt{9x^2 + 6x + 5}} dx = \frac{1}{\sqrt{(3x + 1)^2 + 2^2}} dx$$

$$\int \frac{1}{\sqrt{9x^2 + 6x + 5}} dx = \frac{1}{\sqrt{t^2 + 2^2}} \left(\frac{dt}{3}\right)$$

$$\int \frac{1}{\sqrt{9x^2 + 6x + 5}} dx = \frac{1}{\sqrt{t^2 + 2^2}} dt$$

$$\int \frac{1}{\sqrt{9x^2 + 6x + 5}} dx = \frac{1}{3} \log |t + \sqrt{t^2 + 2^2}| + C$$

$$\int \frac{1}{\sqrt{9x^2 + 6x + 5}} dx = \frac{1}{3} \log |(3x + 1) + \sqrt{(3x + 1)^2 + 2^2}| + C$$

$$\int \frac{1}{\sqrt{9x^2 + 6x + 5}} dx = \frac{1}{3} \log |(3x + 1) + \sqrt{9x^2 + 6x + 5}| + C$$

Where C is an arbitrary constant.

12.

Integrate the function $\dfrac{1}{\sqrt{7-6x-x^2}}$.

Ans -

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} \, dx = \int \frac{1}{\sqrt{7 - (x^2 + 6x)}} \, dx$$

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} \, dx = \int \frac{1}{\sqrt{7 - (x^2 + 6x + 9 - 9)}} \, dx$$

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} \, dx = \int \frac{1}{\sqrt{7 + 9 - (x + 3)^3}} \, dx$$

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} \, dx = \int \frac{1}{\sqrt{16 - (x + 3)^2}} \, dx$$

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} \, dx = \int \frac{1}{\sqrt{4^2 - (x + 3)^2}} \, dx$$

Let $x + 3 = t$

$dx = dt$

Now,

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} \, dx = \int \frac{1}{\sqrt{4^2 - (x + 3)^2}} \, dx$$

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} \, dx = \int \frac{1}{\sqrt{4^2 - t^2}} \, dx$$

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} \, dx = \sin^{-1}\left(\frac{t}{4}\right) + C$$

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} \, dx = \sin^{-1}\left(\frac{x + 3}{4}\right) + C$$

Where C is an arbitrary constant.

13.

Integrate the function $\frac{1}{\sqrt{(x-1)(x-2)}}$.

Ans -

$$\int \frac{1}{\sqrt{(x-1)(x-2)}}\, dx = \int \frac{1}{\sqrt{x^2 - 3x + 2}}\, dx$$

$$\int \frac{1}{\sqrt{(x-1)(x-2)}}\, dx = \int \frac{1}{\sqrt{x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 2}}\, dx$$

$$\int \frac{1}{\sqrt{(x-1)(x-2)}}\, dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{4}\right)}}\, dx$$

$$\int \frac{1}{\sqrt{(x-1)(x-2)}}\, dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dx$$

Now, let $x - \dfrac{3}{2} = t$

$dx = dt$

Now,

$$\int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$\int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \int \frac{1}{\sqrt{(t)^2 - \left(\frac{1}{2}\right)^2}} dt$$

$$\int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \log \left| t + \sqrt{t^2 - \left(\frac{1}{2}\right)^2} \right| + C$$

$$\int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \log \left| \left(x - \frac{3}{2}\right) + \sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{1}{4}} \right| + C$$

$$\int \frac{1}{\sqrt{(x-1)(x-2)}} dx = \log \left| \left(x - \frac{3}{2}\right) + \sqrt{x^2 - 3x + 2} \right| + C$$

Where C is an arbitrary constant.

14.

Integrate the function $\dfrac{1}{\sqrt{8+3x-x^2}}$.

Ans - Let $x - \dfrac{3}{2} = t$

Now, differentiate both sides

$dx = dt$

Now,

$$\int \frac{1}{\sqrt{8 + 3x - x^2}}\,dx = \int \frac{1}{\sqrt{8 - (x^2 - 3x)}}\,dx$$

$$\int \frac{1}{\sqrt{8 + 3x - x^2}}\,dx = \int \frac{1}{\sqrt{8 - \left(x^2 - 3 + \dfrac{9}{4} + \dfrac{9}{4}\right)}}\,dx$$

$$\int \frac{1}{\sqrt{8 + 3x - x^2}}\,dx = \int \frac{1}{\sqrt{8 + \dfrac{9}{4} - \left(x^2 - 3x + \dfrac{9}{4}\right)}}\,dx$$

$$\int \frac{1}{\sqrt{8 + 3x - x^2}}\,dx = \frac{1}{\sqrt{\left(\dfrac{41}{4}\right) - \left(x^2 - \dfrac{3}{2}\right)^2}}\,dx$$

$$\int \frac{1}{\sqrt{8 + 3x - x^2}}\,dx = \frac{1}{\sqrt{\left(\dfrac{\sqrt{41}}{4}\right)^2 - \left(x - \dfrac{3}{2}\right)^2}}\,dx$$

$$\int \frac{1}{\sqrt{8 + 3x - x^2}} dx = \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2 - t^2}} dx$$

$$\int \frac{1}{\sqrt{8 + 3x - x^2}} dx = \sin^{-1} \frac{t}{\frac{\sqrt{41}}{2}} + C$$

$$\int \frac{1}{\sqrt{8 + 3x - x^2}} dx = \sin^{-1} \frac{2\left(x - \frac{3}{2}\right)}{\sqrt{41}} + C$$

$$\int \frac{1}{\sqrt{8 + 3x - x^2}} dx = \sin^{-1} \frac{2x - 3}{\sqrt{41}} + C$$

Where C is an arbitrary constant.

15.

Integrate the function $\frac{1}{\sqrt{(x-a)(x-b)}}$.

Ans - $(x - a)(x - b) = x^2 - (a + b)x + ab$

$$(x - a)(x - b) = x^2 - (a + b)x + \frac{(a + b)^2}{4} - \frac{(a + b)^2}{4} + ab$$

$$(x - a)(x - b) = \left[x - \left(\frac{a + b}{2}\right)\right]^2 - \frac{(a - b)^2}{4}$$

Now, let

$$x - \left(\frac{a + b}{2}\right) = t$$

$$dx = dt$$

Now,

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}\,dx = \int \frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2 - \left(\frac{a-b}{2}\right)^2}}\,dx$$

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}\,dx = \int \frac{1}{\sqrt{t^2 - \left(\frac{a-b}{2}\right)^2}}\,dx$$

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}\,dx = \log\left|t + \sqrt{t^2 - \left(\frac{a-b}{2}\right)^2}\right| + C$$

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}\,dx$$

$$= \log\left|\left\{x - \left(\frac{a+b}{2}\right)\right\} + \sqrt{(x-a)(x-b)}\right| + C$$

Where C is an arbitrary constant.

16.

Integrate the function $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$

Ans - Let $2x^2 + x - 3 = t$

Now, differentiate both sides

$(4x + 1)dx = dt$

$$\int \frac{4x + 1}{\sqrt{2x^2 + x - 3}} dx = \int \frac{1}{\sqrt{t}} dt$$

$$\int \frac{4x + 1}{\sqrt{2x^2 + x - 3}} dx = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2} + 1} + C$$

$$\int \frac{4x + 1}{\sqrt{2x^2 + x - 3}} dx = 2\sqrt{t} + C$$

$$\int \frac{4x + 1}{\sqrt{2x^2 + x - 3}} dx = 2\sqrt{2x^2 + x - 3} + C$$

Where C is an arbitrary constant.

17.

Integrate the function $\frac{x+2}{\sqrt{x^2-1}}$.

Ans –

$$\int \frac{x+2}{\sqrt{x^2-1}}\, dx = \int \frac{x}{\sqrt{x^2-1}}\, dx + \int \frac{2}{\sqrt{x^2-1}}\, dx$$

$$\int \frac{x+2}{\sqrt{x^2-1}}\, dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}\, dx + 2\int \frac{1}{\sqrt{x^2-1}}\, dx \quad \text{...... (1)}$$

Now, for $\dfrac{1}{2}\int \dfrac{2x}{\sqrt{x^2-1}}\, dx$, let $x^2 - 1 = t$

Now, differentiate both sides,

$2x\, dx = dt$

Now, in equation (1),

$$\int \frac{x+2}{\sqrt{x^2-1}}\, dx = \frac{1}{2}\int \frac{2x}{\sqrt{x^2-1}}\, dx + 2\int \frac{1}{\sqrt{x^2-1}}\, dx$$

$$\int \frac{x+2}{\sqrt{x^2-1}}\, dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}} + 2\int \frac{1}{\sqrt{x^2-1}}\, dx$$

$$\int \frac{x+2}{\sqrt{x^2-1}}\, dx = \frac{1}{2}\left(2\sqrt{t}\right) + 2\log\left|x + \sqrt{x^2-1}\right| + C$$

$$\int \frac{x+2}{\sqrt{x^2-1}}\, dx = \sqrt{x^2-1} + 2\log\left|x + \sqrt{x^2-1}\right| + C$$

Where C is an arbitrary constant.

18.

Integrate the function $\dfrac{5x-2}{3x^2+2x+1}$.

Ans -

$$\int \frac{5x - 2}{3x^2 + 2x + 1} dx = \int \frac{5x}{3x^2 + 2x + 1} dx - \int \frac{2}{3x^2 + 2x + 1} dx$$

$$\int \frac{5x - 2}{3x^2 + 2x + 1} dx$$

$$= \frac{5}{6} \int \frac{6x}{3x^2 + 2x + 1} dx - \int \frac{2}{3x^2 + 2x + 1} dx$$

$$\int \frac{5x - 2}{3x^2 + 2x + 1} dx$$

$$= \frac{5}{6} \int \frac{6x + 2 - 2}{3x^2 + 2x + 1} dx - \int \frac{2}{3x^2 + 2x + 1} dx$$

$$\int \frac{5x - 2}{3x^2 + 2x + 1} dx = \frac{5}{6} \int \frac{6x + 2}{3x^2 + 2x + 1} dx$$

$$- \frac{5}{3} \int \frac{1}{3x^2 + 2x + 1} dx - \int \frac{2}{3x^2 + 2x + 1} dx$$

$$\int \frac{5x - 2}{3x^2 + 2x + 1} dx = \frac{5}{6} \int \frac{6x + 2}{3x^2 + 2x + 1} dx$$

$$- \frac{11}{3} \int \frac{1}{3x^2 + 2x + 1} dx \dots \dots (1)$$

Now, for $\dfrac{5}{6} \int \dfrac{6x+2}{3x^2+2x+1} dx$

Let $3x^2 + 2x + 1 = t$

Now, differentiate both sides,

$(6x+2)dx = dt$

Now, in eq (1)

$\int \dfrac{5x-2}{3x^2+2x+1} dx$

$$= \dfrac{5}{6} \int \dfrac{6x+2}{3x^2+2x+1} dx - \dfrac{11}{3} \int \dfrac{1}{3x^2+2x+1} dx$$

$\int \dfrac{5x-2}{3x^2+2x+1} dx = \dfrac{5}{6} \int \dfrac{1}{t} dt - \dfrac{11}{3} \int \dfrac{1}{3\left(x^2+\dfrac{2}{3}x\right)+1} dx$

$$\int \frac{5x-2}{3x^2+2x+1}\,dx$$

$$= \frac{5}{6}\int \frac{1}{t}\,dt - \frac{11}{3}\int \frac{1}{3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)+1}\,dx$$

$$\int \frac{5x-2}{3x^2+2x+1}\,dx = \frac{5}{6}\int \frac{1}{t}\,dt - \frac{11}{3}\int \frac{1}{3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}+1}\,dx$$

$$\int \frac{5x-2}{3x^2+2x+1}\,dx = \frac{5}{6}\int \frac{1}{t}\,dt - \frac{11}{3}\int \frac{1}{3\left(x+\frac{1}{3}\right)^2+\frac{2}{3}}\,dx$$

$$\int \frac{5x-2}{3x^2+2x+1}\,dx = \frac{5}{6}\int \frac{1}{t}\,dt - \frac{11}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}\,dx$$

$$\int \frac{5x-2}{3x^2+2x+1}\,dx = \frac{5}{6}\log|t| - \frac{11}{9}\left(\frac{3}{\sqrt{2}}\tan^{-1}\frac{3x+1}{\sqrt{2}}\right) + C$$

$$\int \frac{5x-2}{3x^2+2x+1}\,dx = \frac{5}{6}\log|3x^2+2x+1|$$

$$-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

Where C is an arbitrary constant.

19.

Integrate the function $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$.

Ans -

$$\int \frac{6x + 7}{\sqrt{(x - 5)(x - 4)}} \, dx = \int \frac{6x + 7}{\sqrt{x^2 - 9x + 20}}$$

$$\int \frac{6x + 7}{\sqrt{(x - 5)(x - 4)}} \, dx = \int \frac{6x}{\sqrt{x^2 - 9x + 20}} \, dx$$

$$+7 \int \frac{1}{\sqrt{x^2 - 9x + 20}} \, dx$$

$$\int \frac{6x + 7}{\sqrt{(x - 5)(x - 4)}} \, dx = 3 \int \frac{2x}{\sqrt{x^2 - 9x + 20}} \, dx$$

$$+7 \int \frac{1}{\sqrt{x^2 - 9x + 20}} \, dx$$

$$\int \frac{6x + 7}{\sqrt{(x - 5)(x - 4)}} \, dx = 3 \int \frac{2x - 9 + 9}{\sqrt{x^2 - 9x + 20}} \, dx$$

$$+7 \int \frac{1}{\sqrt{x^2 - 9x + 20}} \, dx$$

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}} \, d = 3\int \frac{2x-9}{\sqrt{x^2-9x+20}} \, dx$$

$$+\int \frac{27}{\sqrt{x^2-9x+20}} \, dx + 7\int \frac{1}{\sqrt{x^2-9x+20}} \, dx$$

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}} \, dx = 3\int \frac{2x-9}{\sqrt{x^2-9x+20}} \, dx$$

$$+34\int \frac{1}{\sqrt{x^2-9x+20}} \, dx \quad \ldots (1)$$

Now for $3\int \frac{2x-9}{\sqrt{x^2-9x+20}} \, dx$,

Let $x^2 - 9x + 20 = t$,

Differentiate both sides,

$(2x - 9)dx = dt$

Now, from eq. (1),

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx = 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

$$+34\int \frac{1}{\sqrt{x^2-9x+20}}dx$$

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx = 3\int \frac{1}{\sqrt{t}}dt$$

$$+34\int \frac{1}{\sqrt{x^2-9x+\frac{81}{4}-\frac{81}{4}+20}}dx$$

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx = 3\int \frac{1}{\sqrt{t}}dt$$

$$+34\int \frac{1}{\sqrt{x^2-9x+\frac{81}{4}-\frac{81}{4}+20}}dx$$

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}\,dx = 3\int \frac{1}{\sqrt{t}}\,dt$$

$$+34\int \frac{1}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\,dx$$

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}\,dx = 3\left(2\sqrt{t}\right)$$

$$+34\log\left|\left(x-\frac{9}{2}\right) + \sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}\,dx = 3\left(2\sqrt{x^2-9x+20}\right)$$

$$+34\log\left|\left(x-\frac{9}{2}\right) + \sqrt{x^2-9x+20}\right| + C$$

Where C is an arbitrary constant.

20.

Integrate the function $\dfrac{x+2}{\sqrt{4x-x^2}}$.

Ans –

$$\int \frac{x+2}{\sqrt{4x-x^2}} dx = \int \frac{x}{\sqrt{4x-x^2}} dx + 2\int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2}\int \frac{-2x}{\sqrt{4x-x^2}} dx + 2\int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2}\int \frac{4-2x-4}{\sqrt{4x-x^2}} dx + 2\int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}} dx + 2\int \frac{1}{\sqrt{4x-x^2}} dx$$

$$+2\int \frac{1}{\sqrt{4x-x^2}} dx$$

$$\int \frac{x+2}{\sqrt{4x-x^2}} dx = -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}} dx + 4\int \frac{1}{\sqrt{4x-x^2}} dx \quad \dots (1)$$

Now, for $-\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx$,

Let $4x - x^2 = t$,

Now, differentiate both sides,

$(4 - 2x)dx = dt$

From eq. (1) we get,

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx + 4\int \frac{1}{\sqrt{4x-x^2}}dx$$

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\int \frac{1}{\sqrt{t}}dt + 4\int \frac{1}{\sqrt{-(x^2-4x+4-4)}}dx$$

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\int \frac{1}{\sqrt{t}}dt + 4\int \frac{1}{\sqrt{4-(x-2)^2}}dx$$

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}(2\sqrt{t}) + 4\sin^{-1}\frac{x-2}{2} + C$$

Where C is an arbitrary constant.

21.

Integrate the function $\frac{x+2}{\sqrt{x^2+2x+3}}$.

Ans -

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$$

$$= \int \frac{x}{\sqrt{x^2+2x+3}} dx + 2\int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2+2x+3}} dx + 2\int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+2-2}{\sqrt{x^2+2x+3}} dx + 2\int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx - \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$+ 2\int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} d + \int \frac{1}{\sqrt{x^2+2x+3}} dx$$

$$\dots (1)$$

Now, for $\frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$

Let $x^2 + 2x + 3 = t$,

Now, differentiate both sides,

$(2x + 2)dx = dt$

Now, from eq. (1)

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx + \int \frac{1}{\sqrt{x^2+2x+3}}\,dx$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt + \int \frac{1}{\sqrt{x^2+2x+1+2}}\,dx$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt + \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}\,dx$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{1}{2}(2\sqrt{t}) + \log\left|(x+1)+\sqrt{(x+1)^2+2}\right| + C$$

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$$

$$= \sqrt{x^2+2x+3}$$

$$+ \log\left|(x+1)+\sqrt{x^2+2x+3}\right| + C$$

Where C is an arbitrary constant.

22.

Integrate the function $\frac{x+3}{x^2-2x-5}$.

Ans -

$$\int \frac{x+3}{x^2-2x-5}dx = \int \frac{x}{x^2-2x-5}dx + 3\int \frac{1}{x^2-2x-5}dx$$

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\int \frac{2x}{x^2-2x-5}dx + 3\int \frac{1}{x^2-2x-5}dx$$

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\int \frac{2x-2+2}{x^2-2x-5}dx + 3\int \frac{1}{x^2-2x-5}dx$$

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx$$
$$+ 4\int \frac{1}{x^2-2x-5}dx \quad \ldots\ldots(1)$$

Now, for $\frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx$

Let $x^2 - 2x - 5 = t$,

Now, differentiate both sides,

$(2x - 2)dx = dt$

Now, from equation (1),

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx + 4\int \frac{1}{x^2-2x-5}dx$$

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\int \frac{1}{t}dt + 4\int \frac{1}{x^2-2x+1-1-5}dx$$

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\int \frac{1}{t}dt + 4\int \frac{1}{(x-1)^2-(\sqrt{6})^2}dx$$

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log|t| + 4\left(\frac{1}{2\sqrt{6}}\right)\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) + C$$

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log|x^2-2x-5|$$
$$+ 4\left(\frac{1}{2\sqrt{6}}\right)\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) + C$$

Where C is an arbitrary constant.

23.

Integrate the function $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$.

Ans -

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\, dx = \int \frac{5x}{\sqrt{x^2+4x+10}}\, dx$$
$$+ 3\int \frac{1}{\sqrt{x^2+4x+10}}\, dx$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\, dx = \frac{5}{2}\int \frac{2x}{\sqrt{x^2+4x+10}}\, dx$$
$$+ 3\int \frac{1}{\sqrt{x^2+4x+10}}\, dx$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\, dx = \frac{5}{2}\int \frac{2x+4-4}{\sqrt{x^2+4x+10}}\, dx$$
$$+ 3\int \frac{1}{\sqrt{x^2+4x+10}}\, dx$$

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\, dx = \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}\, dx$$
$$- 7\int \frac{1}{\sqrt{x^2+4x+10}}\, dx \quad \text{...... (1)}$$

Now, for $\dfrac{5}{2} \int \dfrac{2x+4}{\sqrt{x^2+4x+10}}\, dx,$

Let $x^2+4x+10 = t,$

Now differentiate both sides,

$(2x+4)\, dx = dt$

Now, from eq. (1),

$$\int \dfrac{5x+3}{\sqrt{x^2+4x+10}}\, dx = \dfrac{5}{2} \int \dfrac{2x+4}{\sqrt{x^2+4x+10}}\, dx$$
$$- 7\int \dfrac{1}{\sqrt{x^2+4x+10}}\, dx$$

$$\int \dfrac{5x+3}{\sqrt{x^2+4x+10}}\, dx = \dfrac{5}{2} \int \dfrac{1}{\sqrt{t}}\, dt - 7\int \dfrac{1}{\sqrt{x^2+4x+4+6}}\, dx$$

$$\int \dfrac{5x+3}{\sqrt{x^2+4x+10}}\, dx = \dfrac{5}{2} \int \dfrac{1}{\sqrt{t}}\, dt - 7\int \dfrac{1}{\sqrt{(x+2)^2 + \left(\sqrt{6}\right)^2}}\, dx$$

$$\int \dfrac{5x+3}{\sqrt{x^2+4x+10}}\, dx = \dfrac{5}{2}\left(2\sqrt{t}\right)$$
$$- 7\log \left| (x+2) + \sqrt{(x+2)^2 + \left(\sqrt{6}\right)^2} \right| + C$$

$$\int \dfrac{5x+3}{\sqrt{x^2+4x+10}}\, dx = 5\sqrt{x^2+4x+10}$$
$$- 7\log \left| (x+2) + \sqrt{x^2+4x+10} \right| + C$$

Where C is an arbitrary constant.

24.

$$\int \frac{dx}{x^2 + 2x + 2} \text{ equals}$$

A. $x\tan^{-1}(x+1) + C$ B. $\tan^{-1}(x+1) + C$

C. $(x+1)\tan^{-1}(x) + C$ D. $\tan^{-1}(x) + C$

Ans –

$$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{x^2 + 2x + 1 + 1}$$

$$\int \frac{dx}{x^2 + 2x + 2} = \int \frac{dx}{(x+1)^1 + 1}$$

$$\int \frac{dx}{x^2 + 2x + 2} = \frac{1}{1}\tan^{-1}\left(\frac{x+1}{1}\right) + C$$

$$\int \frac{dx}{x^2 + 2x + 2} = \tan^{-1}(x+1) + C$$

Thus, the correct answer is B.

25.

$$\int \frac{dx}{\sqrt{9x - 4x^2}} \text{ equals}$$

A. $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$ B. $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{8}\right) + C$

C. $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$ D. $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$

Ans -

$$\int \frac{dx}{\sqrt{9x - 4x^2}} = \int \frac{dx}{\sqrt{-4\left(x^2 - \frac{9}{4}x\right)}}$$

$$\int \frac{dx}{\sqrt{9x - 4x^2}} = \int \frac{1}{\sqrt{-4\left(x^2 - \frac{9}{4}x + \frac{81}{64} - \frac{81}{64}\right)}} dx$$

$$\int \frac{dx}{\sqrt{9x - 4x^2}} = \int \frac{1}{\sqrt{-4\left[\left(x - \frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right]}} dx$$

$$\int \frac{dx}{\sqrt{9x - 4x^2}} = \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2}} dx$$

$$\int \frac{dx}{\sqrt{9x - 4x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x - \frac{9}{8}}{\frac{9}{8}}\right) + C$$

$$\int \frac{dx}{\sqrt{9x - 4x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{8x - 9}{9}\right) + C$$

Thus, the correct answer is B.

Exercise 7.5

1.

Integrate the rational function $\frac{x}{(x+1)(x+2)}$.

Ans - Let $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)}$

$\Rightarrow x = A(x+2) + B(x+1)$

Equating coefficients of x and constant term, we get

$A + B = 1$

$2A + B = 0$

On solving we get $A = -1$ and $B = 2$

$\Rightarrow \dfrac{x}{(x+1)(x+2)} = -\dfrac{1}{(x+1)} + \dfrac{2}{(x+2)}$

$\Rightarrow \displaystyle\int \dfrac{x}{(x+1)(x+2)}\,dx = \int \dfrac{1}{(x+1)} + \dfrac{2}{(x+2)}\,dx$

$= -\log|x+1| + 2\log|x+2| + C$

$= \log(x+2)^2 - \log|x+1| + C$

$= \log\dfrac{(x+2)^2}{(x+1)} + C$

Where C is an arbitrary constant.

2.

Integrate the rational function $\dfrac{1}{x^2-9}$

Ans - Let $\dfrac{1}{(x+3)(x-3)} = \dfrac{A}{(x-3)} + \dfrac{B}{(x-3)}$

Equating coefficients of x and constant term, we get

$A + B = 0$

$-3A + 3B = 1$

On solving, we get A $= -\dfrac{1}{6}$ and B $= \dfrac{1}{6}$

$\Rightarrow \dfrac{1}{(x+3)(x-3)} = \dfrac{-1}{6(x+3)} + \dfrac{1}{6(x-3)}$

$\Rightarrow \int \dfrac{1}{(x^2-9)} dx = \int \left(\dfrac{-1}{6(x+3)} + \dfrac{1}{6(x-3)} \right) dx$

$= -\dfrac{1}{6}\log|x+3| + \dfrac{1}{6}\log|x-3| + C$

$= \dfrac{1}{6}\log\dfrac{|(x-3)|}{|(x+3)|} + C$

Where C is an arbitrary constant.

3.

Integrate the rational function $\dfrac{3x-1}{(x-1)(x-2)(x-3)}$

Ans –

Let $\dfrac{3x-1}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$

$3x - 1 = A(x-2)(x-3) + B(x-1)(x-3)$
$\qquad\qquad + C(x-1)(x-2)$

Equating coefficients of x^2, x and constant term, we get,

$A + B + C = 0$

$-5A - 4B - 3C = 3$

$6A + 3B + 2C = -1$

Solving these equations, we obtain $A = 1$ $B = -5$,and $C = 4$

$\dfrac{3x-1}{(x-1)(x-2)(x-3)} = \dfrac{1}{(x-1)} - \dfrac{5}{(x-2)} + \dfrac{4}{(x-3)}$

$\Rightarrow \int \dfrac{3x-1}{(x-1)(x-2)(x-3)} dx$

$\qquad = \int \left\{ \dfrac{1}{(x-1)} - \dfrac{5}{(x-2)} + \dfrac{4}{(x-3)} \right\} dx$

$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$

Where C is an arbitrary constant.

4.

Integrate the rational function $\dfrac{x}{(x-1)(x-2)(x-3)}$

Ans -

$$\frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Equating coefficients of x^2, x and constant term, we get,

$$A + B + C = 0$$

$$45 - 3C = 1$$

$$6A + 4B + 2C = 0$$

Solving these equations, we obtain $A = \frac{1}{2}$, $B = 2$ and $C = \frac{3}{2}$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)} dx$$

$$= \int \left\{ \frac{1}{2(x-1)} - \frac{2}{(x-2)} + \frac{3}{2(x-3)} \right\} dx$$

$$= \frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$$

Where C is an arbitrary constant.

5.

Integrate the rational function $\frac{2x}{x^2+3x+2}$

Ans –

Let $\dfrac{2x}{x^2 + 3x + 2} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)}$

$2x = A(x+2) + B(x+1)$ .... (1)

Equating coefficients of x^2, x and constant term, we get,

$A + B = 2$

$2A + B = 0$

Solve these equations, we obtain $A = -2$ and $B = 4$

$\therefore \dfrac{2x}{(x+1)(x+2)} = \dfrac{-2}{(x+1)} + \dfrac{4}{(x+2)}$

$\Rightarrow \int \dfrac{2x}{(x+1)(x+2)} dx = \int \left\{ \dfrac{4}{(x+2)} - \dfrac{2}{(x+1)} \right\} dx$

$= 4\log|x+2| - 2\log|x+1| + C$

Where C is an arbitrary constant.

6.

Integrate the rational function $\dfrac{1-x^2}{x(1-2x)}$

Ans - Let $\dfrac{x}{(x^2+1)(x-1)} = \dfrac{Ax+B}{(x^2+1)} + \dfrac{C}{(x-1)}$

$x = (Ax+B)(x-1) + C(x^2+1)$

$x = Ax^2 - Ax + Bx - B + Cx^2 + C$

Equating coefficients of x^2, x, and constant term, we get

$A - A + B = 1$

$-B + C = 0$

On solving these equations, we get $A = -\dfrac{1}{2}, B = \dfrac{1}{2}, C = \dfrac{1}{2}$

$\therefore \dfrac{x}{(x^2+1)(x-1)} = \dfrac{\left(-\dfrac{1}{2}x + \dfrac{1}{2}\right)}{x^2+1} + \dfrac{\dfrac{1}{2}}{(x-1)}$

$\Rightarrow \displaystyle\int \dfrac{x}{(x^2+1)(x-1)} = -\dfrac{1}{2}\int \dfrac{x}{x^2+1}dx + \dfrac{1}{2}\int \dfrac{1}{x^2+1}dx$

$$+ \dfrac{1}{2}\int \dfrac{1}{x-1}dx$$

$= -\dfrac{1}{4}\displaystyle\int \dfrac{2x}{x^2+1}dx + \dfrac{1}{2}\tan^{-1}x + \dfrac{1}{2}\log|x-1| + C$

Solving these equation, we obtain $A = 2$ and $B = 3$

$$\therefore \frac{2 - x}{x(1 - 2x)} = \frac{2}{x} + \frac{3}{1 - 2x}$$

Substituting in equation (1), we get

$$\frac{1 - x^2}{x(1 - 2x)} = \frac{1}{2} + \frac{1}{2}\left\{\frac{2}{x} + \frac{3}{(1 - 2x)}\right\}$$

$$\int \frac{1 - x^2}{x(1 - 2x)} dx = \int \left\{\frac{1}{2} + \frac{1}{2}\left(\frac{2}{x} + \frac{3}{(1 - 2x)}\right)\right\} dx$$

$$= \frac{x}{2} + \log|x| + \frac{3}{2(-2)}\log|1 - 2x| + C$$

$$= \frac{x}{2} + \log|x| - \frac{3}{4}\log|1 - 2x| + c$$

Where C is an arbitrary constant.

7.

Integrate the rational function $\dfrac{x}{(x^2 + 1)(x - 1)}$

Ans - Let $\dfrac{x}{(x^2 + 1)(x - 1)} = \dfrac{Ax + B}{(x^2 + 1)} + \dfrac{C}{(x - 1)}$

$x = (Ax + B)(x - 1) + C(x^2 + 1)$

$x = Ax^2 - Ax + Bx - B + Cx^2 + C$

Equating coefficients of x^2, x, and constant term, we get

$A - A + B = 1$

$-B + C = 0$

On solving these equations, we get $A = -\dfrac{1}{2}$, $B = \dfrac{1}{2}$, $C = \dfrac{1}{2}$

$\therefore \dfrac{x}{(x^2 + 1)(x - 1)} = \dfrac{\left(-\dfrac{1}{2}x + \dfrac{1}{2}\right)}{x^2 + 1} + \dfrac{\dfrac{1}{2}}{(x - 1)}$

$\Rightarrow \displaystyle\int \dfrac{x}{(x^2 + 1)(x - 1)} = -\dfrac{1}{2}\displaystyle\int \dfrac{x}{x^2 + 1}dx + \dfrac{1}{2}\displaystyle\int \dfrac{1}{x^2 + 1}dx$

$$+ \dfrac{1}{2}\displaystyle\int \dfrac{1}{x - 1}dx$$

$= -\dfrac{1}{4}\displaystyle\int \dfrac{2x}{x^2 + 1}dx + \dfrac{1}{2}\tan^{-1}x + \dfrac{1}{2}\log|x - 1| + C$

Consider $\displaystyle\int \dfrac{2x}{x^2 + 1}dx$, let $(x^2 + 1) = t \Rightarrow 2xdx = dt$

$\Rightarrow \displaystyle\int \dfrac{2x}{x^2 + 1}dx - \displaystyle\int \dfrac{dt}{t} - \log|t| - \log|x^2 + 1|$

$\therefore \displaystyle\int \dfrac{x}{(x^2 + 1)(x - 1)} = -\dfrac{1}{4}\log|x^2 + 1| + \dfrac{1}{2}\tan^{-1}x$

$$+ \dfrac{1}{2}\log|x - 1| + C$$

Where C is an arbitrary constant.

8.

Integrate the rational function $\dfrac{x}{(x-1)^2(x+2)}$

Ans - Let $\dfrac{x}{(x-1)^2(x+2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x+2)}$

$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

Equating coefficients of x^2, x and constant term, we get

$A + C = 0$

$A + B - 2C = 1$

$-2A + 2B + C = 0$

On solving. we get $A = \dfrac{2}{9}, B = \dfrac{1}{3}$ and $C = \dfrac{-2}{9}$

$\therefore \dfrac{x}{(x-1)^2(x+2)} = \dfrac{2}{9(x-1)} + \dfrac{1}{3(x-1)^2} - \dfrac{2}{9(x+2)}$

$\Rightarrow \int \dfrac{x}{(x-1)^2(x+2)} dx$

$= \dfrac{2}{9} \int \dfrac{1}{(x-1)} dx + \dfrac{1}{3} \int \dfrac{1}{(x-1)^2} dx$

$\qquad - \dfrac{2}{9} \int \dfrac{1}{(x+2)} dx$

$= \dfrac{2}{9} \log|x-1| + \dfrac{1}{3}\left(\dfrac{-1}{x-1}\right) - \dfrac{2}{9}\log|x+2| + C$

$= \dfrac{2}{9} \log\left|\dfrac{x-1}{x+2}\right| - \dfrac{1}{3(x-1)} + C$

Where C is an arbitrary constant.

9.

Integrate the rational function $\dfrac{3x+5}{x^3 - x^2 - x + 1}$

Ans - $\dfrac{3x+5}{x^3-x^2-x+1} = \dfrac{3x+5}{(x-1)^2(x+1)}$

Let $\dfrac{3x+5}{(x-1)^2(x+1)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x+1)}$

$3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$

$3x+5 = A(x^2-1) + B(x+1) + C(x^2+1-2x)$

Equating coefficient of x^2, x and constant term, we get

$A + C = 0$

$B - 2C - 3$

$-A + B + C = 5$

On solving. we get $A = -\dfrac{1}{2}$, $B = 4$ and $C = \dfrac{1}{2}$

$\therefore \dfrac{3x+5}{(x-1)^2(x+1)} = \dfrac{-1}{2(x-1)} + \dfrac{4}{(x-1)^2} + \dfrac{1}{2(x+1)}$

$\Rightarrow \displaystyle\int \dfrac{3x+5}{(x-1)^2(x+1)}\, dx$

$$= -\dfrac{1}{2}\int \dfrac{1}{x-1}\, dx + 4\int \dfrac{1}{(x-1)^2}\, dx$$

$$+ \dfrac{1}{2}\int \dfrac{1}{(x+1)}\, dx$$

$$= -\dfrac{1}{2}\log|x-1| + 4\left(\dfrac{-1}{x-1}\right) + \dfrac{1}{2}\log|x+1| + C$$

$$= \dfrac{1}{2}\log\left|\dfrac{x+1}{x-1}\right| - \dfrac{4}{(x-1)} + C$$

Where C is an arbitrary constant.

10.

Integrate the rational function $\dfrac{2x-3}{(x^2-1)(2x+3)}$

Ans -

$$\frac{2x - 3}{(x^2 - 1)(2x + 3)} = \frac{2x - 3}{(x + 1)(x - 1)(2x + 3)}$$

Let $\dfrac{2x - 3}{(x + 1)(x - 1)(2x + 3)} = \dfrac{A}{(x + 1)} + \dfrac{B}{(x - 1)} + \dfrac{C}{(2x + 3)}$

$\Rightarrow (2x - 3) = A(x - 1)(2x + 3) + B(x + 1)(2x + 3)$
$\qquad\qquad + C(x + 1)(x - 1)$

$\Rightarrow (2x - 3) = A(2x^2 + x - 3) + B(2x^2 + 5x + 3)$
$\qquad\qquad + C(x^2 - 1)$

$\Rightarrow (2x - 3) = (2A + 2B + C)x^2 + (A + 5B)x$
$\qquad\qquad + (-3A + 3B - C)$

Equating coefficients of x^2, x and constant, we get

$2A + 2B + C = 0$

$A + 5B = 2$

$-3A + 3B - C = -3$

On solving, we get $A = \dfrac{5}{2}$, $B = -\dfrac{1}{10}$, and $C = -\dfrac{24}{5}$

$\therefore \dfrac{2x - 3}{(x + 1)(x - 1)(2x + 3)}$

$\qquad = \dfrac{5}{2(x + 1)} - \dfrac{1}{10(x - 1)} - \dfrac{24}{5(2x + 3)}$

$\Rightarrow \displaystyle\int \dfrac{2x - 3}{(x^2 - 1)(2x + 3)}\, dx$

$\qquad = \dfrac{5}{2}\displaystyle\int \dfrac{1}{(x + 1)}\, dx - \dfrac{1}{10}\displaystyle\int \dfrac{1}{x - 1}\, dx$

$\qquad\quad - \dfrac{24}{5}\displaystyle\int \dfrac{1}{(2x + 3)}\, dx$

$= \dfrac{5}{2}\log|x + 1| - \dfrac{1}{10}\log|x - 1| - \dfrac{24}{5 \times 2}\log|2x + 3|$

$= \dfrac{5}{2}\log|x + 1| - \dfrac{1}{10}\log|x - 1| - \dfrac{12}{5}\log|2x + 3| + C$

Where C is an arbitrary constant

11.

Integrate the rational function $\dfrac{5x}{(x + 1)(x^2 - 4)}$

Ans $-\dfrac{5x}{(x+1)(x^2-4)} = \dfrac{5x}{(x+1)(x+2)(x-2)}$

Let $\dfrac{5x}{(x+1)(x+2)(x-2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x-2)}$

$5x = A(x+2)(x-2) + B(x+1)(x-2) + C(x+1)(x+2)$

Equating coefficients of x^2, x and constant, we get

$A + B + C = 0$

$B + 3C = 5$

$4A - 2B + 2C = 0$

On solving, we obtain $A - \dfrac{5}{3}, B - -\dfrac{5}{2},$ and $C - \dfrac{5}{6}$

$\therefore \dfrac{5x}{(x+1)(x+2)(x-2)} = \dfrac{5}{3(x+1)} \pm \dfrac{5}{2(x+2)} + \dfrac{5}{6(x-2)}$

$\Rightarrow \displaystyle\int \dfrac{5x}{(x+1)(x^2-4)}\,dx$

$\qquad = \dfrac{5}{3}\displaystyle\int \dfrac{1}{(x+1)}\,dx - \dfrac{5}{2}\displaystyle\int \dfrac{1}{(x+2)}\,dx$

$\qquad + \dfrac{5}{6}\displaystyle\int \dfrac{1}{(x-2)}\,dx$

$= \dfrac{5}{3}\log|x+1| - \dfrac{5}{2}\log|x+2| + \dfrac{5}{6}\log|x-2| + C$

Where C is an arbitrary constant.

12.

Integrate the rational function $\dfrac{x^3+x+1}{x^2-1}$

Ans - It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(x^3 + x + 1)$ by $x^2 - 1$, we get

$$\frac{x^3 + x + 1}{x^2 - 1} = x + \frac{2x + 1}{x^2 - 1}$$

Let $\dfrac{2x + 1}{x^2 - 1} = \dfrac{A}{(x + 1)} + \dfrac{B}{(x - 1)}$

$2x + 1 = A(x - 1) + B(x + 1)$

Equating coefficients of x and constant, we get,

$A + B = 2$

$-A + B = 1$

On solving. we obtain $A - \dfrac{1}{2}$ and $B - \dfrac{3}{2}$

$$\therefore \frac{x^3 + x + 1}{x^2 - 1} = x + \frac{1}{2(x + 1)} + \frac{3}{2(x - 1)}$$

$$\Rightarrow \int \frac{x^3 + x + 1}{x^2 + 1} dx = \int x \, dx + \frac{1}{2} \int \frac{1}{(x + 1)} dx + \frac{3}{2} \int \frac{1}{(x - 1)} dx$$

$$= \frac{x^2}{2} + \log|x + 1| + \frac{3}{2} \log|x - 1| + C$$

Where C is an arbitrary constant.

13.

Ans -

Let $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{(1-x)} + \dfrac{Bx+C}{(1+x^2)}$

$2 = A(1+x^2) + (Bx+C)(1-x)$

$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$

Equating coefficient of x^2, x, and constant term, we get

$A - B = 0$

$B - C = 0$

$A + C = 2$

On solving these equations, we get, $A = 1, B = 1$, and $C = 1$

$\therefore \dfrac{2}{(1-x)(1+x^2)} = \dfrac{1}{1-x} + \dfrac{x+1}{1+x^2}$

$\Rightarrow \int \dfrac{2}{(1-x)(1+x^2)} dx$

$$= \int \dfrac{1}{1-x} dx + \int \dfrac{x}{1+x^2} dx + \int \dfrac{1}{1+x^2} dx$$

$$= -\int \dfrac{1}{1-x} dx + \dfrac{1}{2} \int \dfrac{2x}{1+x^2} dx + \int \dfrac{1}{1+x^2} dx$$

$$= -\log|x - 1| + \dfrac{1}{2}\log|1 + x^2| + \tan^{-1}x + C$$

Where C is an arbitrary constant.

14.

Integrate the rational function $\dfrac{3x-1}{(x+2)^2}$

Ans –

Let $\dfrac{3x - 1}{(x + 2)^2} = \dfrac{A}{(x + 2)} + \dfrac{B}{(x + 2)^2}$

$\Rightarrow 3x - 1 = A(x + 2) + B$

Equating coefficient of x and constant term, we get

$A = 3$

$2A + B = -1$

On solving these equations, we get, $A = 3, B = -7$

$\therefore \dfrac{3x - 1}{(x + 2)^2} = \dfrac{3}{(x + 2)} - \dfrac{7}{(x + 2)^2}$

$\Rightarrow \int \dfrac{3x - 1}{(x + 2)^2} dx = 3\int \dfrac{1}{(x + 2)} dx - 7\int \dfrac{1}{(x + 2)^2} dx$

$= 3\log|x + 2| - 7\left(\dfrac{-1}{(x + 2)}\right) + C$

$= 3\log|x + 2| + \dfrac{7}{(x + 2)} + C$

Where C is an arbitrary constant

15.

Integrate the rational function $\dfrac{1}{x^4 - 1}$

Ans -

$$\frac{1}{(x^4-1)} - \frac{1}{(x^2-1)(x^2+1)} - \frac{1}{(x+1)(x-1)(1+x^2)}$$

Let $\dfrac{1}{(x+1)(x-1)(1+x^2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-1)} + \dfrac{Cx+D}{(x^2+1)}$

$1 = A(x-1)(1+x^2) + B(x+1)(1+x^2)$
$\qquad + (Cx+D)(x^2-1)$

$1 = A(x^3 + x - x^2 - 1) + B(x^3 + x + x^2 + 1) + Cx^3 + Dx^2$
$\qquad - Cx - D$

$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x$
$\qquad + (-A+B-D)$

Equating coefficient of x^3, x^2, x and constant term we get,

$A + B + C - 0$

$-A + B + D = 0$

$A + B - C = 0$

$-A + B - D = 1$

On solving these equations we get,

$A = -\dfrac{1}{4}, \ B = \dfrac{1}{4}, C = 0, \text{ and } D = -\dfrac{1}{2}$

$\therefore \dfrac{1}{(x^n-1)} = \dfrac{-1}{4(x+1)} + \dfrac{1}{4(x-1)} + \dfrac{1}{2(x^2+1)}$

$\Rightarrow \int \dfrac{1}{x^4-1} dx - -\dfrac{1}{4}\log|x-1| + \dfrac{1}{4}\log|x-1| - \dfrac{1}{2}\tan^1 x + C$

$= \dfrac{1}{4}\log\left|\dfrac{x-1}{x+1}\right| - \dfrac{1}{2}\tan^1 x + C$

Where C is an arbitrary constant.

16.

Integrate the rational function $\dfrac{1}{x(x^n+1)}$

[Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$.]

Ans - Multiplying numerator and denominator by x^{n-1} we obtain

$$\frac{1}{x(x^n + 1)} = \frac{x^{n-1}}{x^{n-1}x(x^n + 1)} = \frac{x^{n-1}}{x^n(x^n + 1)}$$

Let $x^n = t \Rightarrow nx^{-1}dx = dt$

$$\therefore \int \frac{1}{x(x^n + 1)}dx = \int \frac{x^{n-1}}{x^7(x^n + 1)}dx = \frac{1}{n}\int \frac{1}{t(t + 1)}dt$$

Let $\dfrac{1}{t(t + 1)} = \dfrac{A}{t} + \dfrac{B}{(t + 1)}$

$1 = A(1 + t) + Bt$

Equating coefficient of t and constant, we get

$A = 1$ and $B = -1$

$$\frac{1}{t(t + 1)} = \frac{1}{t} - \frac{1}{(1 + t)}$$

$$\Rightarrow \int \frac{1}{x(x^4 + 1)}dx = \frac{1}{n}\int \left\{\frac{1}{t} - \frac{1}{(1 + t)}\right\}dx$$

$$\frac{1}{n}[\log|t| - \log|t + 1|] + C$$

$$= \frac{1}{n}[\log|x^n| - \log|x^n + 1|] + C$$

Where C is an arbitrary constant

17.

Integrate the rational function $\dfrac{\cos x}{(1-\sin x)(2-\sin x)}$

[Hint: Put $\sin x = t$]

Ans - Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore \int \frac{\cos x}{(1 - \sin x)(2 - \sin x)}\, dx = \int \frac{dt}{(1 - t)(2 - t)}$$

Let $\dfrac{1}{(1 - t)(2 - t)} = \dfrac{A}{(1 - t)} + \dfrac{B}{(2 - t)}$

$1 = A(2 - t) + B(1 - t)$

Equating coefficients of t and constant, we get

$-A - B = 0$

$2A + B = 1$

On solving these equations, we obtain $A = 1$ and $B = -1$

$$\therefore \frac{1}{(1 - t)(2 - t)} = \frac{1}{(1 - t)} - \frac{1}{(2 - t)}$$

$$\Rightarrow \int \frac{\cos x}{(1 - \sin x)(2 - \sin x)}\, dx = \int \left\{ \frac{1}{1 - t} - \frac{1}{(2 - t)} \right\} dt$$

$$= \log \left| \frac{2 - t}{1 - t} \right| + C$$

$$= \log \left| \frac{2 - \sin x}{1 - \sin x} \right| + C$$

Where C is an arbitrary constant

18.

Integrate the rational function $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Ans –

$$\frac{(x^2 + 1)(x^2 + 2)}{(x^2 + 3)(x^2 + 4)} = 1 - \frac{4x^2 + 10}{(x^2 + 3)(x^2 + 4)}$$

Let $\dfrac{(4x^2 + 10)}{(x^2 + 3)(x^2 + 4)} = \dfrac{Ax + B}{(x^2 + 3)} + \dfrac{Cx + D}{(x^2 + 4)}$

$4x^2 + 10 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 3)$

Equating coefficients of x^3, x^2, x and constant term, we get,

$B + D = 4$

$4A + 3C = 0$

$4B + 3D = 10$

On solving these equations, we obtain

$A = 0, B = -2, C = 0$ and $D = 6$

$\therefore \dfrac{(4x^2 + 10)}{(x^2 + 3)(x^2 + 4)} = \dfrac{-2}{(x^2 + 3)} + \dfrac{6}{(x^2 + 4)}$

$\dfrac{(x^2 + 1)(x^2 + 2)}{(x^2 + 3)(x^2 + 4)} = 1 - \left(\dfrac{-2}{(x^2 + 3)} + \dfrac{6}{(x^2 + 4)}\right)$

$\Rightarrow \int \dfrac{(x^2 + 1)(x^2 + 2)}{(x^2 + 3)(x^2 + 4)} dx = \int \left\{1 + \dfrac{2}{(x^2 + 3)} - \dfrac{6}{(x^2 + 4)}\right\} dx$

$= \int \left\{1 + \dfrac{2}{x^2 + (\sqrt{3})^2} - \dfrac{6}{x^2 + 2^2}\right\}$

$= x + 2\left(\dfrac{1}{\sqrt{3}} \tan^{-1} \dfrac{x}{\sqrt{3}}\right) - 6\left(\dfrac{1}{2} \tan^{-1} \dfrac{x}{2}\right) + C$

$= x + \dfrac{2}{\sqrt{3}} \tan^{-1} \dfrac{x}{\sqrt{3}} - 3\tan^{-1} \dfrac{x}{2} + C$

Where C is an arbitrary constant

19.

Integrate the rational function $\dfrac{2x}{(x^2+1)(x^2+3)}$

Ans – Let $x^2 = t \Rightarrow 2x\,dx = dt$

$\therefore \displaystyle\int \frac{2x}{(x^2+1)(x^2+3)}\,dx = \int \frac{dt}{(t+1)(t+3)}$

Let $\dfrac{1}{(t+1)(t+3)} = \dfrac{A}{(t+1)} + \dfrac{B}{(t+3)}$

$1 = A(t+3) + B(t+1)$

Equating coefficients of t and constant, we get

$A + B = 0$

$3A + B = 1$

On solving these equations we get $A = \dfrac{1}{2}$ and $B = -\dfrac{1}{2}$

$\therefore \dfrac{1}{(t+1)(t+3)} = \dfrac{1}{2(t+1)} - \dfrac{1}{2(t+3)}$

$\Rightarrow \displaystyle\int \frac{2x}{(x^2+1)(x^2+3)}\,dx = \int \left\{ \frac{1}{2(t+1)} - \frac{1}{2(t+3)} \right\} dt$

$= \dfrac{1}{2}\log|(t+1)| - \dfrac{1}{2}\log|t+3| + C$

$= \dfrac{1}{2}\log\left|\dfrac{t+1}{t+3}\right| + C$

$= \dfrac{1}{2}\log\left|\dfrac{x^2+1}{x^2+3}\right| + C$

Where C is an arbitrary constant.

20.

Integrate the rational function $\dfrac{1}{x(x^4-1)}$

Ans - Multiplying numerator and denominator by x^3 we get

$$\frac{1}{x(x^4-1)} = \frac{x^3}{x^4(x^4-1)}$$

$$\therefore \int \frac{1}{x(x^4-1)}dx = \int \frac{x^3}{x^4(x^4-1)}dx$$

Let $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\therefore \int \frac{1}{x(x^4-1)}dx = \frac{1}{4}\int \frac{dt}{t(t-1)}$$

Let $\dfrac{1}{t(t-1)} = \dfrac{A}{t} + \dfrac{B}{(t-1)}$

$$1 = A(t-1) + Bt$$

Equating coefficient of t and constant, we get

$$A + B = 0$$

$$-A = 1$$

On solving these equations we get $A = -1$ and $B = 1$

$$\Rightarrow \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{x(x^4-1)}dx = \frac{1}{4}\int \left\{\frac{-1}{t} + \frac{1}{t-1}\right\}dt$$

$$-\frac{1}{4}\left[-\log|t| + \log|t-1|\right] + C$$

$$= \frac{1}{4}\log\left|\frac{t-1}{t}\right| + C$$

$$= \frac{1}{4}\log\left|\frac{x^4-1|}{x^4}\right| + C$$

Where C is an arbitrary constant

21.

Integrate the rational function $\dfrac{1}{e^x - 1}$

[Hint: put $\dfrac{1}{e^x - 1}$]

Ans - Let $e^x = t \Rightarrow e^x \, dx = dt$

$$\Rightarrow \int \frac{1}{t(t-1)} = \frac{A}{t} = \frac{B}{t-1}$$

$1 = A(t-1) + Bt$

Equating coefficients of t and constant, we get

$A + B = 0$

$-A = 1$

On solving these equations we get $A = -1$ and $B = 1$

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} = \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{t(t-1)} = \log \left| \frac{t-1}{t} \right| + C$$

$$\Rightarrow \log \left| \frac{e^x - 1}{2} \right| + C$$

Where C is an arbitrary constant.

22.

$\int \dfrac{x \, dx}{(x-1)(x-2)}$ equals

(A) $A \log \left| \dfrac{(x-1)^2}{x-2} \right| + C$

(B) $\log \left| \left(\dfrac{x-1}{x-2} \right)^2 \right| + C$

(C) $\log \left| \dfrac{(x-1)^2}{x-2} \right| + C$

(D) $\log |(x-1)(x-2)| + C$

Ans -

Let $\dfrac{xdx}{(x-1)(x-2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)}$

Equating coefficients of x and constant, we get

$A + B = 1$

$-2A - B = 0$

On solving these equations we get $A = -1$ and $B = 2$

$\dfrac{xdx}{(x-1)(x-2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)}$

$\Rightarrow \displaystyle\int \dfrac{x}{(x-1)(x-2)} dx = \int \left\{ \dfrac{-1}{(x-1)} + \dfrac{2}{(x-2)} \right\} dx$

$= -\log|x-1| + 2\log|x-2| + C$

$= \log \left| \dfrac{(x-1)^2}{x-2} \right| + C$

Hence. the correct answer is **B**

23.

$\displaystyle\int \dfrac{dx}{x(x^2+1)}$ equals

(A) $\log|x| - \dfrac{1}{2}\log(x^2 + 1) + C$

(B) $\log|x| + \dfrac{1}{2} \log(x^2 + 1) + C$

(C) $-\log|x| - \dfrac{1}{2}\log(x^2 + 1) + C$

(D) $\dfrac{1}{2}\log|x| + \log(x^2 + 1) + C$

Ans -

Let $\dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$1 = A(x^2 + 1) + (Bx + C)x$

Equating coefficients of x^2, x and constant term, we get

$A + B = 0$

$C = 0$

$A = 1$

On solving these equations, we get $A = 1, B = -1, C = 0$

$\therefore \dfrac{1}{x(x^2 + 1)} = \dfrac{1}{x} + \dfrac{-x}{x^2 + 1}$

$\Rightarrow \int \dfrac{1}{x(x^2 + 1)}\, dx = \int \left\{ \dfrac{1}{x} - \dfrac{x}{x^2 + 1} \right\} dx$

$= \log|x| - \dfrac{1}{2}\log|x^2 + 1| + C$

Where C is an arbitrary constant

Hence the correct answer is **A**.

Exercise 7.6

1.

Integrate the function x sin x

Ans - Let $I = \int x \sin x \, dx$

Consider $u = x$ and $v = \sin x$ and integrating by parts, to obtain

$$I = \int x \sin x \, dx - \int \left(\frac{d}{dx}x\right)\{\sin \int x \, dx\}dx$$

$$= x(-\cos x) - \int 1.(-\cos x)dx$$

$$= -x\cos x + \sin x + C$$

2.

Integrate the function x sin 3x

Ans - Let $I = \int x \sin 3x \, dx$

Consider $u = x$ and $v = \sin 3x$

Integrating by parts, we get

$$I = x\int \sin 3x \, dx - \int \left\{\left(\frac{d}{dx}x\right)\int \sin 3x \, dx\right\}$$

$$= x\left(\frac{-\cos 3x}{3}\right) - \int 1 \cdot \left(\frac{-\cos 3x}{3}\right)dx$$

$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x dx = \frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$$

3.

Integrate the function $x^2 e^x$

Ans - Let $I = \int x^2 e^x dx$

Consider $u = x^2$ and $v = e^x$

$I = x^2 \int e^x dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx$

$= x^2 e^x - \int 2x - e^x dx$

$= x^2 e^x - 2 \int x \cdot e^x dx$

Again using integration by parts, we get

$= x^2 e^x - 2 \left[x \cdot \int e^x dx - \int \left\{ \left(\frac{d}{dx} x^2 \right) \int e^x dx \right\} dx \right]$

$= x^2 e^x - 2 \left[xe^x - \int e^x dx \right]$

Simplifying,

$= x^2 e^x - 2[xe^x - e^x]$

$= x^2 e^x - 2xe^x + 2e^x + C$

$= e^x(x^2 - 2x + 2) + C$

4.

Integrate the function x log x

Ans - Let $I = \int x \log x \, dx$

Consider $u = \log x$ and $v = x$ and integrating by parts, we get

$I = \log x \int x dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x dx \right\} dx$

$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$

$= \frac{x^2 \log x}{2} \cdot \sqrt{\frac{x}{2}} dx = \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$

5.

Integrate the function x log 2x

Ans - Let $I = \int x \log 2x \, dx$

Consider $u = \log 2x$ and $v = x$ and integrating by parts, to obtain.

$$I = \log 2x \int x \, dx - \int \left\{ \left(\frac{d}{dx} 2\log x \right) \int x \, dx \right\} dx$$

$$= \log 2x \cdot \frac{x^2}{2} - \int \frac{2}{2x} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2 \log 2x}{2} - \int \frac{x}{2} \, dx$$

Integrating using the power rule

$$I = \frac{x^2 \log 2x}{2} - \frac{x^2}{4} + C$$

6.

Integrate the function $x^2 \log x$

Ans - Let $I = \int x^2 \log x \, dx$

Consider $u = \log 2$ and $v = x^2$ and integrating by parts, to obtain

$$I = \log x \int x^2 dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x^2 dx \right\} dx$$

$$= \log x \left(\frac{x^3}{3} \right) - \int \frac{1}{x} \cdot \frac{x^3}{3} dx$$

Integrating using the power rule

$$I = \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$$

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

7.

Integrate the function x sin⁻¹x

Ans - Let $I = \int x \sin^{-1} x \, dx$

Consider $u = \sin^{-1} x$ and $v = x$

Integrating by parts, we get

$$I = \sin^{-1} x \int x \, dx \int \left\{ \left(\frac{d}{dx} \sin^{-1} x \right) \int x \, dx \right\} dx$$

$$= \sin^{-1} x \left(\frac{x^2}{2} \right) - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

Adding and subtracting by 1

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \right\} dx$$

Simplifying we get,

$$I = \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left\{ \sqrt{1 - x^2} - \frac{1}{\sqrt{1 - x^2}} \right\} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \int \sqrt{1 - x^2} dx - \int \frac{1}{\sqrt{1 - x^2}} dx \right\}$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \left\{ \frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right\} + C$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{x}{4} \sqrt{1 - x^2} + \frac{1}{4} \sin^{-1} x - \frac{1}{2} \sin^{-1} x + C$$

$$= \frac{1}{4} (2x^2 + 1) in^{-1} x + \frac{x}{4} \sqrt{1 - x^2} + C$$

8.

Integrate the function $x \tan^{-1} x$

Ans - Let $I = \int x\tan^{-1}x \, dx$

Consider $u = \tan^{-1}x$ and $v = x$

Integrating by parts, we get

$$I = \tan^{-1}x \int x \, dx - \int \left\{ \left(\frac{d}{dx}\tan^{-1}x \right) \int x \, dx \right\} dx$$

$$= \tan^{-1}x \left(\frac{x^2}{2} \right) \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2} \, dx$$

Adding and subtracting by -1

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \left(\frac{x^2+1}{1+x^2} - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2\tan^{-1}x}{2} - \frac{1}{2}(x - \tan^{-1}x) + C$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C$$

9.

Integrate the function $x\cos^{-1}x$

Ans - Let $I = \int x\cos^{-1}x\,dx$

Taking $u = \cos^{-1}x$, $v = x$ and integrating by parts, we get,

$$I = \cos^{-1}x\int x\,dx - \int \left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int x\,dx\right\}dx$$

$$= \cos^{-1}x\frac{x^2}{2} - \int \frac{-1}{\sqrt{1-x^2}}\cdot\frac{x^2}{2}\,dx$$

Adding and subtracting by -1

$$= \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\int \left\{\sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$$

$$= \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}\int \sqrt{1-x^2}\,dx - \frac{1}{2}\int \left(\frac{-1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^2\cos^{-1}x}{2} - \frac{1}{2}I_1 - \frac{1}{2}\cos^{-1}x \quad \ldots\ldots(1)$$

Where $I_1 = \int \sqrt{1-x^2}\, dx$

$\Rightarrow I_1 = x\int \sqrt{1-x^2} - \int \frac{d}{dx}\sqrt{1-x^2}\int x dx$

$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}\, dx$

$\Rightarrow I_1 = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\, dx$

$\Rightarrow I_1 = x\sqrt{1-x^2} - \left\{\int \sqrt{1-x^2}\, dx + \int \frac{-dx}{\sqrt{1-x^2}}\, dx\right\}$

$\Rightarrow I_1 = x\sqrt{1-x^2} - \{I_1 + \cos^{-1}x\}$

$\Rightarrow 2I_1 = x\sqrt{1-x^2} - \cos^{-1}x$

$\therefore I_1 = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x$

Substituting in (1), we get

$I = \frac{x\cos^{-1}x}{2} - \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x\right) - \frac{1}{2}\cos^{-1}x$

$= \frac{(2x^2-1)}{4}\cos^{-1}x - \frac{x}{4}\sqrt{1-x^2} + C$

10.

Integrate the function $(\sin^{-1}x)^2$

Ans - Let $I = \int (\sin^{-1}x)^2 \cdot 1dx$

Consider $u = \sin^{-1}x$, $v = 1$ and integrating by parts, we get

$$I = \int (\sin^{-1}x) \cdot \int 1dx - \int \left\{\frac{d}{dx}(\sin^{-1}x)^2 \cdot \int 1. dx\right\} dx$$

$$= (\sin^{-1}x)^2 x - \int \frac{2\sin^{-1}x}{\sqrt{1-x^2}} \cdot xdx$$

$$= x(\sin^{-1}x)^2 + \int \sin^{-1}x \cdot \left(\frac{-2x}{\sqrt{1-x^2}}\right) dx$$

$$= x(\sin^{-1}x)^2 + \left[\sin^{-1}x\int \frac{-2x}{\sqrt{1-x^2}} dx\right.$$

$$\left. - \int \left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int \frac{-2x}{\sqrt{1-x^2}} dx\right\} dx\right]$$

$$= x(\sin^{-1}x)^2 + \left[\sin^{-1}x \cdot 2\sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}}\right.$$

$$\left. \cdot 2\sqrt{1-x^2}dx\right]$$

$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - \int 2dx$$

$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + C$$

11.

Integrate the function $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Ans - Let $I = \int \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

Multiplying and dividing by 2

$I = \dfrac{-1}{2}\int \dfrac{-2x}{\sqrt{1-x^2}} \cdot \cos^{-1}x\,dx$

Consider $u = \cos^{-1}x$ and $v = \left(\dfrac{-2x}{\sqrt{1-x^2}}\right)$ and integrating by parts, we get

$I = \dfrac{-1}{2}\left[\cos^{-1}x\int \dfrac{-2x}{\sqrt{1-x^2}}dx\right.$

$\left. - \int \left\{\left(\dfrac{d}{dx}\cos^{-1}x\right)\int \dfrac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$

$= \dfrac{-1}{2}\left[\cos^{-1}x \cdot 2\sqrt{1-x^2} - \int \dfrac{-1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}dx\right]$

$= \dfrac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + \int 2dx\right]$

$= \dfrac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + 2x\right] + C$

$= -\left[\sqrt{1-x^2}\cos^{-1}x + x\right] + C$

12.

Integrate the function x sec²x

Ans - Let $I = \int x \sec^2 x\, dx$

Consider $u = x$ and $v = \sec^2 x$ and integrating by parts, we get

$$I = x\int \sec^2 x\, dx - \int \left\{ \left\{ \frac{d}{dx} x \right\} \int \sec^2 x\, dx \right\} dx$$

$$= x \tan x - \int 1 \cdot \tan x\, dx$$

$$= x \tan x + \log|\cos x| + C$$

13.

Integrate the function $\tan^{-1} x$

Ans - Let $I = \int \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}\,dx$

Multiplying and dividing by 2

$I = \dfrac{-1}{2}\int \dfrac{-2x}{\sqrt{1-x^2}} \cdot \cos^{-1}x\,dx$

Consider $u = \cos^{-1}x$ and $v = \left(\dfrac{-2x}{\sqrt{1-x^2}}\right)$ and integrating by parts, we get

$I = \dfrac{-1}{2}\left[\cos^{-1}x \int \dfrac{-2x}{\sqrt{1-x^2}}\,dx \right.$

$\left. - \int \left\{\left(\dfrac{d}{dx}\cos^{-1}x\right)\int \dfrac{-2x}{\sqrt{1-x^2}}\,dx\right\}dx\right]$

$= \dfrac{-1}{2}\left[\cos^{-1}x \cdot 2\sqrt{1-x^2} - \int \dfrac{-1}{\sqrt{1-x^2}} \cdot 2\sqrt{1-x^2}\,dx\right]$

$= \dfrac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + \int 2\,dx\right]$

$= \dfrac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x + 2x\right] + C$

$= -\left[\sqrt{1-x^2}\cos^{-1}x + x\right] + C$

14.

Integrate the function $x(\log x)^2 dx$

Ans - $I = \int x(\log x)^2 dx$

Consider $u = (\log x)^2$, $v = 1$ and integrating by parts, we get

$$I = (\log)^2 \int x dx - \int \left[\left\{ \left(\frac{d}{dx} \log x \right)^2 \right\} \int x dx \right] dx$$

$$= \frac{x^2}{2} (\log x)^2 - \left[\int 2\log x \cdot \frac{1}{x} \cdot \frac{x^2}{2} dx \right]$$

$$= \frac{x^2}{2} (\log x)^2 - \int x \log x dx$$

Again, using Integration by parts, to obtain

$$I = \frac{x^2}{2} (\log x)^2 - \left[\log x \int x dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int x dx \right\} dx \right]$$

$$= \frac{x^2}{2} (\log x)^2 - \left[\frac{x^2}{2} - \log x - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \right]$$

$$= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{1}{2} \int x dx$$

$$= \frac{x^2}{2} (\log x)^2 - \frac{x^2}{2} \log x + \frac{x^2}{4} + C$$

15.

Integrate the function $(x^2 + 1)\log x$

Ans - Let $I = \int (x^2 + 1)\log x\, dx$

$= \int x^2 \log x\, dx + \int \log x\, dx$

Let $I = I_1 + I_2$ ... (1)

Where, $I_1 = \int x^2 \log x\, dx$ and $I_2 = \int \log x\, dx$

$I_1 = \int x^2 \log x\, dx$

Consider $u = \log x$, $v = x^2$ and integrating by parts, we get

$I_1 = \log x - \int x^2 dx - \int \left\{ \left(\dfrac{d}{dx}\log x\right) \int x^2 dx \right\} dx$

$= \log x \cdot \dfrac{x^3}{3} - \int \dfrac{1}{x} \cdot \dfrac{x^3}{3} dx$

$= \dfrac{x^3}{3}\log x - \dfrac{1}{3}\int x^2 dx$

$= \dfrac{x^3}{3}\log x - \dfrac{x^3}{9} + C_1$ ... ... (2)

$I_2 = \int \log x\, dx$

Consider u=log x and v = 1 and integrating by parts, to obtain

$$I_2 = \log x \int 1.\,dx - \int \left\{ \left(\frac{d}{dx} \log x \right) \int 1.\,dx \right\}$$

$$= \log x \cdot x - \int \frac{1}{x} x\,dx$$

$$= x \log x - x.. \,(3)$$

Using equations (2) and (3) in (1), we get

$$I = \frac{x^3}{3} \log x - \frac{x^3}{9} + C_1 + x\log x - x + C_2$$

$$= \frac{x^3}{3} \log x - \frac{x^3}{9} + x\log x - x + (C_1 + C_2)$$

$$= \left(\frac{x^3}{3} + x \right) \log x - \frac{x^3}{9} - x + C$$

16.

Integrate the function $e^x(\sin x + \cos x)$

Ans - Consider $I = \int e^x(\sin x + \cos x)dx$

Consider $f(x) = \sin x$

$f'(x) = \cos x$

$I = \int e^x\{f(x) + f'(x)\}dx$

Since, $\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + C$

$\therefore I = e^x \sin x + C$

17.

Integrate the function $\dfrac{xe^x}{(1+x)^2}$

Ans - Consider $I = \int \dfrac{xe^x}{(1+x)^2} dx = \int e^x \left\{\dfrac{x}{(1+x)^2}\right\} dx$

$= \int e^x \left\{\dfrac{1+x-1}{(1+x)^2}\right\} dx = \int e^x \left\{\dfrac{1}{1+x} - \dfrac{1}{(1+x)^2}\right\} dx$

Here, $f(x) = \dfrac{1}{1+x}$ $f'(x) = -\dfrac{1}{(1+x)^2}$

$\Rightarrow \int \dfrac{xe^x}{(1+x)^2} dx = \int e^x \{f(x) + f'(x)dx\} + C$

Since, $\int e^x \{f(x) + f'(x)dx\} = e^x f(x) + C$

$\Rightarrow \therefore \int \dfrac{xe^x}{(1+x)^2} dx = \int \dfrac{e^x}{1+x} + C$

18.

Integrate the function - $e^x \left(\dfrac{1+\sin x}{1+\cos x}\right)$

Ans - First simplify $-e^x \left(\frac{1+\sin x}{1+\cos x}\right)$

We know that,

$$1 + \sin x = \sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}$$

$$1 + \cos x = 2\cos^2 \frac{x}{2}$$

$$\therefore e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) = e^x \left(\frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}{2\cos^2 \frac{x}{2}}\right)$$

$$= e^x \left(\frac{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2}{2\cos^2 \frac{x}{2}}\right)$$

$$= \frac{1}{2} e^x \left(\frac{\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2}{\cos^2 \frac{x}{2}}\right)$$

$$= \frac{1}{2} e^x \left(\frac{\sin \frac{x}{2} + \cos \frac{x}{2}}{\cos \frac{x}{2}}\right)^2$$

$$= \frac{1}{2}e^x \left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} + \frac{\cos \frac{x}{2}}{\cos \frac{x}{2}} \right)^2$$

$$= \frac{1}{2}e^x \left(\tan \frac{x}{2} + 1 \right)^2$$

$$= \frac{1}{2}e^x \left(\tan^2 \frac{x}{2} + 1 + 2\tan \frac{x}{2} \right)$$

But, $1 + \tan^2 \frac{x}{2} = \sec^2 \frac{x}{2}$

$$= \frac{1}{2}e^x \left(\sec^2 \frac{x}{2} + 2\tan \frac{x}{2} \right)$$

$$= e^x \left(\frac{1}{2}\sec^2 \frac{x}{2} + \tan \frac{x}{2} \right)$$

$$\Rightarrow e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) = e^x \left(\frac{1}{2}\sec^2 \frac{x}{2} + \tan \frac{x}{2} \right)$$

It is known that, $\int e^x \{f(x) + f'(x)\}dx = e^x f(x) + C$

If we say, $f(x) = \tan \frac{x}{2} \Rightarrow f'(x) = \frac{1}{2}\sec^2 \frac{x}{2}$

Thus, we get $-\int e^x \left(\frac{1+\sin x}{1+\cos x} \right) dx = e^x \tan \frac{x}{2} + C$

19.

Integrate the function $e^x \left(\frac{1}{x} - \frac{1}{x^2} \right)$

Ans - Let $I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$

Suppose, $f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2}$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \frac{e^x}{x} + C$

20.

Integrate the function $\dfrac{(x-3)e^x}{(x-1)^3}$

Ans - $\int e^x \dfrac{(x-3)}{(x-1)^3} dx = \int e^x \left[\dfrac{(x-1-2)}{(x-1)^3} \right] dx$

$= \int e^x \left[\dfrac{(x-1)}{(x-1)^3} - \dfrac{2}{(x-1)^3} \right] dx$

$= \int e^x \left[\dfrac{1}{(x-1)^2} - \dfrac{2}{(x-1)^3} \right] dx$

Suppose, $f(x) = \dfrac{1}{(x-1)^2} \Rightarrow f'(x) = -\dfrac{2}{(x-1)^3}$

It is known that, $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$

$\therefore \int e^x \dfrac{(x-3)}{(x-1)^3} dx = \dfrac{e^x}{(x-1)^2} + C$

21.

Integrate the function $e^{2x} \sin x$

Ans - Say, $I = \int e^{2x} \sin x \, dx$

Using Integration by parts,

$\int uvdx = u\int vdx - \int (u'\int vdx)dx$

With $u = \sin x$, $v = e^{2x}$

$I = \int e^{2x}\sin x dx = \sin x\int e^{2x}dx - \int \left[\left(\frac{d}{dx}\sin x\right)\int e^{2x}dx\right]dx$

$= \sin x \frac{e^{2x}}{2} - \int \left[(\cos x)\frac{e^{2x}}{2}\right]dx$

$= \sin x \frac{e^{2x}}{2} - \frac{1}{2}\int (e^{2x}\cos x)dx$

Perform Integration by parts for $-\int (e^{2x}\cos x)dx$

$= \sin x \frac{e^{2x}}{2} - \frac{1}{2}\left\{\cos x\int e^{2x}dx - \int \left[\left(\frac{d}{dx}\cos x\right)\int e^{2x}dx\right]dx\right\}$

$= \sin x \frac{e^{2x}}{2} - \frac{1}{2}\left\{\cos x\frac{e^{2x}}{2} - \int \left[(-\sin x)\frac{e^{2x}}{2}\right]dx\right\}$

$= \sin x \frac{e^{2x}}{2} - \frac{1}{2}\left\{\cos x\frac{e^{2x}}{2} + \frac{1}{2}\int (\sin x)e^{2x}dx\right\}$

$$= \sin x \frac{e^{2x}}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}\{\int (\sin x)e^{2x}dx\}$$

But, $I = \int e^{2x}\sin x dx$

$$\Rightarrow I = \sin x \frac{e^{2x}}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}I$$

$$\Rightarrow I + \frac{1}{4}I = \sin x \frac{e^{2x}}{2} - \frac{e^{2x}\cos x}{4}$$

$$\Rightarrow \frac{5}{4}I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}$$

$$\Rightarrow \frac{5}{4}I = \frac{2e^{2x}\sin x}{4} - \frac{e^{2x}\cos x}{4}$$

$$5I = e^{2x}(2\sin x - \cos x)$$

$$\therefore I = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

22.

Integrate the function $\sin^{-1}\left(\frac{2x}{1+x^3}\right)$

Ans – Let $x = \tan\theta \Rightarrow dx = \sec^2\theta d\theta$

$\therefore \sin^{-1}\left(\dfrac{2x}{1+x^3}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^3\theta}\right)$

But, $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^3\theta}$

$\Rightarrow \sin^{-1}\left(\dfrac{2x}{1+x^3}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^3\theta}\right)$

$\Rightarrow \sin^{-1}\left(\dfrac{2x}{1+x^3}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

$\therefore \int \sin^{-1}\left(\dfrac{2x}{1+x^3}\right) dx = \int 2\theta\sec^2\theta d\theta$

$= 2\int \theta\sec^2\theta \, d\theta$

Perform Integration by parts,

$\int uv dx = u\int vdx - \int (u'\int vdx)dx$

With $- u = \theta \; v = \sec^2\theta$

$2\int \theta\sec^2\theta d\theta = 2\left\{\theta\int \sec^2\theta d\theta - \int \left[\left(\dfrac{d}{d\theta}\theta\right)\int \sec^2\theta d\theta\right] d\theta\right\}$

$= 2\{\theta\tan\theta - \int [\tan\theta]d\theta\}$

$= 2\{\theta\tan\theta - (-\log|\cos\theta|)\} + C$

$= 2\{\theta\tan\theta + \log|\cos\theta|\} + C$

Replace $\theta = \tan^{-1}x$

$= 2\{\tan^{-1}x\tan(\tan^{-1}x) + \log|\cos(\tan^{-1}x)|\} + C$

It is known that $- \tan^{-1}x = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}}$

$$= 2\left\{\tan^{-1}x(x) + \log\left|\cos\left(\cos^{-1}\dfrac{1}{\sqrt{1+x^2}}\right)\right|\right\} + C$$

$$= 2\left\{x\tan^{-1}x + \log\left|\dfrac{1}{\sqrt{1+x^2}}\right|\right\} + C$$

$$= 2\left\{x\tan^{-1}x + \log(1+x^2)^{-\frac{1}{2}}\right\} + C$$

Here, $\log m^n = n\log m$

$$= 2\left\{x\tan^{-1}x - \dfrac{1}{2}\log(1+x^2)\right\} + C$$

$$= 2x\tan^{-1}x - \log(1+x^2) + C$$

$$\therefore \int \sin^{-1}\left(\dfrac{2x}{1+x^3}\right)dx = 2x\tan^{-1}x - \log(1+x^2) + C$$

23.

Choose the correct answer: $\int x^2 e^{x^3}dx$ **equals**

(A) $\dfrac{1}{3}e^{x^3} + C$　　　　　　　　(B) $\dfrac{1}{3}e^{x^2} + C$

(C) $\dfrac{1}{2}e^{x^3} + C$　　　　　　　　(D) $\dfrac{1}{2}e^{x^2} + C$

Ans - Let $I = \int x^2 e^{x^3} dx$

Suppose, $t = x^3 \Rightarrow dt = 3x^2 dx$

Rewriting the equation,

$I = \int x^2 e^{x^3} dx = \dfrac{1}{3} \int e^t dt$

$\Rightarrow I = \dfrac{1}{3} \int e^t dt = \dfrac{1}{3} e^t + C$

Replacing $t = x^3$

$\Rightarrow I = \dfrac{1}{3} e^{x^3} + C$

Hence, correct option is **A**.

24.

Choose the correct answer: $\int e^x secx(1 + tanx)dx$

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

Ans - Let $I = \int e^x secx(1 + tanx)$ dx

$\Rightarrow I = \int e^x (secx + secxtanx)dx$

Suppose, $f(x) = secx$

$\Rightarrow f'(x) = secx\ tanx$

It is known that, $\int e^x \{f(x) + f'(x)\}dx = e^x f(x) + C$

$\Rightarrow I = \int e^x (secx + secxtanx)dx$

$= e^x secx + C$

Hence, correct option is **B**.

Exercise 7.7

1.

Integrate the function $\sqrt{4-x^2}$.

Ans - Let $I = \int \sqrt{4-x^2}\,dx = \int \sqrt{(2)^2 - (x)^2}\,dx$

It is known that,

$$\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

Substitute a = 2 and solve the integral

$$\therefore I = \frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} + C$$

$$= \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + C$$

Where C is an arbitrary constant.

2.

Integrate the function $\sqrt{1-4x^2}$

Ans - Let $I = \int \sqrt{1 - 4x^2}\,dx$

$= \int \sqrt{(1)^2 - (2x)^2}\,dx$

Let $2x = t \Rightarrow 2dx = dt$

$\therefore I = \dfrac{1}{2}\int \sqrt{(1)^2 - (t)^2}$

It is known that

$$\sqrt{a^2 - x^2}\,dx = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} + C$$

Substitute $a = 1$ and $t = 2x$.

$\therefore I = \dfrac{1}{2}\left[\dfrac{t}{2}\sqrt{1 - t^2} + \dfrac{1}{2}\sin^{-1}t\right] + C$

$= \dfrac{t}{4}\sqrt{1 - t^2} + \dfrac{1}{4}\sin^{-1}t + C$

$= \dfrac{2x}{4}\sqrt{1 - 4x^2} + \dfrac{1}{4}\sin^{-1}2x + C$

$= \dfrac{x}{2}\sqrt{1 - 4x^2} + \dfrac{1}{4}\sin^{-1}2x + C$

Where C is an arbitrary constant

3.

Integrate the function $\sqrt{x^2 + 4x + 6}$

Ans - Let $I = \int \sqrt{x^2 + 4x + 6} \, dx$

$I = \int \sqrt{x^2 + 4x + 4 + 2} \, dx$

$I = \int \sqrt{(x^2 + 4x + 4) + 2} \, dx$

$I = \int \sqrt{(x + 2)^2 + (\sqrt{2})^2} \, dx$

It is known that

$$\sqrt{x^2 + a^2} \, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$$

Substitute $a = \sqrt{2}$ and $x = x + 2$.

$$\therefore I = \frac{(x + 2)}{2}\sqrt{x^2 + 4x + 6} + \frac{2}{2}\log\left|(x + 2) + \sqrt{x^2 + 4x + 6}\right|$$
$$+ C$$

$$= \frac{(x + 2)}{2}\sqrt{x^2 + 4x + 6} + \log\left|(x + 2) + \sqrt{x^2 + 4x + 6}\right| + C$$

Where C is an arbitrary constant.

4.

Integrate the function $\sqrt{x^2 + 4x + 1}$

Ans - Let $I = \int \sqrt{x^2 + 4x + 1} \, dx$

$I = \int \sqrt{(x^2 + 4x + 4) - 3} \, dx$

$I = \int \sqrt{(x + 2)^2 - (\sqrt{3})^2} \, dx$

It is known that

$$\sqrt{x^2 - a^2} \, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Substitute $a = \sqrt{3}$ and $x = x + 2$.

$$\therefore I = \frac{(x + 2)}{2}\sqrt{x^2 + 4x + 1} - \frac{3}{2}\log\left|(x + 2) + \sqrt{x^2 + 4x + 1}\right|$$
$$+ C$$

Where C is an arbitrary constant.

5.

Integrate the function $\sqrt{1 - 4x - x^2}$.

Ans - Let $I = \int \sqrt{1 - 4x - x^2}\,dx$

$I = \int \sqrt{1 - (x^2 + 4x + 4 - 4)}\,dx$

$I = \int \sqrt{1 + 4 - (x + 2)^2}\,dx$

$I = \int \sqrt{\left(\sqrt{5}\right)^2 - (x + 2)^2}\,dx$

It is known that,

$$\sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

Substitute $a = \sqrt{5}$ and $x = x + 2$.

$$\therefore I = \frac{(x + 2)}{2}\sqrt{1 - 4x - x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x + 2}{\sqrt{5}}\right) + C$$

Where C is an arbitrary constant.

6.

Integrate the function $\sqrt{x^2 + 4x - 5}$

Ans - Let $I = \int \sqrt{x^2 + 4x - 5}\,dx$

$I = \int \sqrt{(x^2 + 4x + 4) - 9}\,dx$

$I = \int \sqrt{(x + 2)^2 - (3)^2}\,dx$

It is known that

$$\sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Substitute $a = 3$ and and $x = x + 2$.

$$\therefore I = \frac{(x + 2)}{2}\sqrt{x^2 + 4x - 5} - \frac{9}{2}\log|(x + 2) + \sqrt{x^2 + 4x - 5}|$$
$$+ C$$

Where C is an arbitrary constant.

7.

Integrate the function $\sqrt{1 + 3x - x^2}$

Ans - Let $I = \int \sqrt{1 + 3x - x^2}\,dx$

$$I = \int \sqrt{1 - \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right)}\,dx$$

$$I = \int \sqrt{\left(1 + \frac{9}{4}\right) - \left(x - \frac{3}{2}\right)^2}\,dx$$

$$I = \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}\,dx$$

It is known that

$$\sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

Substitute $a = \frac{\sqrt{13}}{2}$ and $x = x - \frac{3}{2}$

$$\therefore I = \frac{\left(x - \frac{3}{2}\right)}{2}\sqrt{1 - 3x - x^2} + \frac{13}{4 \times 2}\sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + C$$

$$= \frac{2x - 3}{4}\sqrt{1 - 3x - x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x - 3}{\sqrt{13}}\right) + C$$

Where C is an arbitrary constant.

8.

Integrate the function $\sqrt{x^2 + 3x}$

Ans - Let I $= \int \sqrt{x^2 + 3x}\,dx$

$$I = \int \sqrt{x^2 + 3x + \frac{9}{4} - \frac{9}{4}}\,dx$$

$$I = \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$$

It is known that,

$$\sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Substitute $a = \frac{3}{2}$ and $x = x + \frac{3}{2}$.

$$\therefore I = \frac{x + \frac{3}{2}}{2}\sqrt{x^2 + 3x} - \frac{\frac{9}{4}}{2}\log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x}\right| + C$$

$$= \frac{(2x + 3)}{2}\sqrt{x^2 + 3x} - \frac{9}{8}\log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x}\right| + C$$

Where C is an arbitrary constant.

9.

Integrate the function $\sqrt{1 + \frac{x^2}{9}}$

Ans - Let $I = \int \sqrt{1 + \dfrac{x^2}{9}}\,dx$

$I = \dfrac{1}{3} \int \sqrt{9 + x^2}\,dx$

$I = \dfrac{1}{3} \int \sqrt{(3)^2 + (x)^2}\,dx$

It is known that

$$\sqrt{x^2 + a^2}\,dx = \dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$$

Substitute $a = 3$,

$$\therefore I = \dfrac{1}{3}\left[\dfrac{x}{2}\sqrt{x^2 + 9} + \dfrac{9}{2}\log\left|x + \sqrt{x^2 + 9}\right|\right] + C$$

$$= \dfrac{x}{6}\sqrt{x^2 + 9} + \dfrac{3}{2}\log\left|x + \sqrt{x^2 + 9}\right| + C$$

Where C is an arbitrary constant.

10.

$\int \sqrt{1 + x^2}$ is equals to

(A) $\dfrac{x}{2}\sqrt{1 + x^2} + \dfrac{1}{2}\log\left|x + \sqrt{1 + x^2}\right| + C$

(B) $\dfrac{2}{3}(1 + x^2)^{\frac{2}{3}} + C$

(C) $\dfrac{2}{3}x(1 + x^2)^{\frac{2}{3}} + C$

(D) $\dfrac{x^2}{2}\sqrt{1 + x^2} + \dfrac{1}{2}x^2\log\left|x + \sqrt{1 + x^2}\right| + C$

Ans - It is known that

$$\sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$$

Let $I = \int \sqrt{1 + x^2}\,dx$

Substitute $a = 1$

$$\therefore \int \sqrt{1 + x^2}\,dx = \frac{x}{2}\sqrt{1 + x^2} + \frac{1}{2}\log\left|x + \sqrt{1 + x^2}\right| + C$$

Hence, correct answer is **A**.

11.

$\int \sqrt{x^2 - 8x + 7}$ is equal to

A. $\dfrac{1}{2}(x - 4)\sqrt{x^2 - 8x + 7} + 9\log\left|x - 4 + \sqrt{x^2 - 8x + 7}\right| + C$

B. $\dfrac{1}{2}(x + 4)\sqrt{x^2 - 8x + 7} + 9\log|x + 4 + \sqrt{x^2 - 8x + 7}| + C$

C. $\dfrac{1}{2}(x - 4)\sqrt{x^2 - 8x + 7} - 3\sqrt{2}\log|x - 4 + \sqrt{x^2 - 8x + 7}| + C$

D. $\dfrac{1}{2}(x - 4)\sqrt{x^2 - 8x + 7} - \dfrac{9}{2}\log|x - 4 + \sqrt{x^2 - 8x + 7}| + C$

Ans - Let $I = \int \sqrt{x^2 - 8x + 7}dx$

$I = \int \sqrt{(x^2 - 8x + 16) - 9}dx$

$I = \int \sqrt{(x - 4)^2 - (3)^2}dx$

It is known that

$$\sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Substitute a = 3 and x = x − 4.

$$\therefore I = \frac{(x - 4)}{2}\sqrt{x^2 - 8x + 7} - \frac{9}{2}\log|(x - 4) + \sqrt{x^2 - 8x + 7|}$$
$$+ C$$

Hence, correct answer is **D.**

Exercise 7.8

1.

Evaluate the definite integral $\int_{-1}^{1}(x + 1)dx$

Ans - $I = \int\limits_{-1}^{1} (x + 1)dx$

$\int (x + 1)dx = \dfrac{x^2}{2} + x$

Suppose the function $F(x) = \dfrac{x^2}{2} + x$

By second fundamental theorem of calculus, we get

$I = F(1) = F(-1)$

$= \left(\dfrac{1}{2} + 1\right) - \left(\dfrac{1}{2} - 1\right)$

$= \dfrac{1}{2} + 1 - \dfrac{1}{2} + 1$

$= 2$

2.

Evaluate the definite integral $\int_{2}^{3} \dfrac{1}{x} dx$.

Ans - $I = \int_{2}^{3} \dfrac{1}{x} dx$

Thus, $\int \dfrac{1}{x} dx = \log|x|$

Suppose the function is $F(x) = \log|x|$

By second fundamental theorem of calculus, we get

$I = F(3) - F(2)$

$= \log|3| - \log|2|$

$= \log \dfrac{3}{2}$

3.

Evaluate the definite integral $\int_1^2 (4x^3 - 5x^2 + 6x + 9)\,dx$

Ans - To simplify the question let's say,

$$I = \int_1^2 (4x^3 - 5x^2 + 6x + 9)\,dx$$

$$= 4\left(\frac{x^4}{4}\right) - 5\left(\frac{x^3}{4}\right) + 6\left(\frac{x^2}{4}\right) + 9(x)$$

Again, let's say the function is

$$\Rightarrow x^4 - \frac{5x^3}{3} + 3x^2 + 9x = F(x)$$

By second fundamental theorem of calculus, we get

$$I = F(2) - F(1)$$

$$I = \left\{2^4 - \frac{5(2)^3}{3} + 2(2)^2 + 9(2)\right\}$$

$$- \left\{(1)^4 - \frac{5(1)^3}{3} + 3(1)^2 + 9(1)\right\}$$

$$= \left(16 - \frac{40}{3} + 12 + 18\right) - \left(1 - \frac{5}{3} + +3 + 9\right)$$

$$= 16 - \frac{40}{3} + 12 + 18 - 1 + \frac{5}{3} - 3 - 9$$

$$= 33 - \frac{35}{3}$$

$$= \frac{99 - 35}{3}$$

$$= \frac{64}{3}$$

4.

Evaluate the definite integral $\int_0^{\frac{\pi}{4}} (\sin 2x\, dx)$

Ans - To simplify the question let's say

$$I = \int_0^{\frac{\pi}{4}} (\sin 2x \, dx)$$

$$\int \sin 2x \, dx = \left(\frac{-\cos 2x}{2} \right)$$

Again, let's say function is $\left(\frac{-\cos 2x}{2} \right) = F(x)$

By second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4} \right) - F(0)$$

$$= -\frac{1}{2} \left[\cos 2\left(\frac{\pi}{4} \right) - \cos 0 \right]$$

$$= -\frac{1}{2} \left[\cos 2\left(\frac{\pi}{4} \right) - \cos 0 \right]$$

$$= -\frac{1}{2} \left[\cos 2\left(\frac{\pi}{4} \right) - \cos 0 \right]$$

$$= -\frac{1}{2} [0 - 1]$$

$$= \frac{1}{2}$$

5.

Evaluate the definite integral $\int_0^{\frac{\pi}{2}} (\cos 2x \, dx)$

Ans - To simplify the question let's say

$$I = \int_0^{\frac{\pi}{2}} (\cos 2x \, dx)$$

$$\int \cos 2x \, dx = \left(\frac{-\sin 2x}{2}\right)$$

Again, let's say function is $\left(\frac{\sin 2x}{2}\right) = F(x)$

By second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{2}\right) - F(0)$$

$$= \frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right) - \sin 0\right]$$

$$= \frac{1}{2}[\sin \pi - \sin 0]$$

$$= \frac{1}{2}[0 - 0]$$

$$= 0$$

6.

Evaluate the definite integral $\int_4^5 e^x \, dx$

Ans - To simplify the question let's say,

$$I = \int_4^5 e^x \, dx$$

$$\int_4^5 e^x \, dx = e^x = F(x)$$

By second fundamental theorem of calculus, we get.

$$I = F(5) - F(4)$$

$$= e^5 - e^4 = e^4(e - 1)$$

7.

Evaluate the definite integral $\int_0^{\frac{\pi}{2}} \tan x \, dx$

Ans - To simplify the question let's say,

$$I = \int_0^{\frac{\pi}{2}} \tan x \, dx$$

$$\int \tan dx = -\log|\cos x|$$

Again, let's say the function $|s - \log|\cos x| = F(x)$

By second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|$$

$$= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$$

$$= -\log(2)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\log 2$$

8.

Evaluate the definite integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x \, dx$

Ans - To simplify the question let's say,

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cosec x \, dx$$

$$\int \cosec x \, dx = \log \left| \cosec x \frac{\pi}{6} - \cot \frac{\pi}{6} \right|$$

Again, let's say the function is $F(x) = \log |\cos x|$

By second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right)$$

$$= \log \left| \cosec \frac{\pi}{4} - \cot \frac{\pi}{4} \right| - \log \left| \cosec \frac{\pi}{6} - \cot \frac{\pi}{6} \right|$$

$$= \log \left| \sqrt{2} - 1 \right| - \log |2 - \sqrt{3}|$$

$$= \log \left(\frac{\sqrt{2} - 1}{2 - \sqrt{3}} \right)$$

9.

Evaluate the definite integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

Ans - To simplify the question let's say,

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

Again, let's say the function is

$$I = \int \frac{dx}{\sqrt{1-x^2}} = F(x)$$

$$\sin^{-1} x - F(x)$$

By second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

10.

Evaluate the definite integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}}$

Ans - To simplify the question let's say,

$$I = \int_0^1 \frac{dx}{\sqrt{1+x^2}}$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \tan^{-1}x$$

Again, let's say the function is

$$\tan^{-1}x = F(x)$$

By second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$

$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

11.

Evaluate the definite integral $\int_2^3 \frac{dx}{x^2 - 1}$

Ans - To simplify the question let's say,

$$I = \int_2^3 \frac{dx}{x^2 - 1}$$

$$\int \frac{dx}{x^2 - 1} = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right|$$

Again, let's say the function is

$$\frac{1}{2} \log \left| \frac{x-1}{x+1} \right| = F(x)$$

By second fundamental theorem of calculus, we get

$$I = F(3) - F(2)$$

$$= \frac{1}{2} \left[\log \left| \frac{3-1}{3+1} \right| - \log \left| \frac{2-1}{2+1} \right| \right]$$

$$= \frac{1}{2} \left[\log \left| \frac{2}{4} \right| - \log \left| \frac{1}{3} \right| \right]$$

$$= \frac{1}{2} \left[\log \frac{1}{2} - \log \frac{1}{3} \right]$$

$$= \frac{1}{2} \left[\log \frac{3}{2} \right]$$

12.

Evaluate the definite integral $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

Ans - To simplify the question let's say,

$$\int \cos^2 x \, dx = \int \left(\frac{1 + \cos x}{2}\right) dx$$

$$\frac{x}{2} + \frac{\sin 2x}{4} = \frac{1}{2}\left(x + \frac{\sin 2x}{4}\right)$$

Again, let's say the function is

$$\frac{1}{2}\left(x + \frac{\sin 2x}{4}\right) = F(x)$$

By second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{2}\right) - F(0)$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0 - 0 + 0\right]$$

$$= \frac{\pi}{4}$$

13.

Evaluate the definite integral $\int_2^3 \frac{x\,dx}{x^2+1}$

Ans - To simplify the question let's say

$$I = \int\limits_{2}^{3} \frac{xdx}{x^2 + 1}$$

$$\int \frac{xdx}{x^2 + 1} = \frac{1}{2} \int \frac{2x}{x^2 + 1} dx = \frac{1}{2} \log(1 + x^2)$$

Again, let's say the function is

$$\frac{1}{2} \log(1 + x^2) = F(x)$$

By second fundamental theorem of calculus, we get

$$I = F(3) - F(2)$$

$$= \frac{1}{2} [\log(1 + (3)^2) - \log(1 + (2)^2)]$$

$$= \frac{1}{2} [\log 10 - \log 5]$$

$$= \frac{1}{2} \left[\log \frac{10}{5} \right]$$

$$= \frac{1}{2} [\log 2]$$

14.

Evaluate the definite integral $\int_0^1 \frac{2x+3}{5x^2+1} dx$

Ans - To simplify the question let's say,

$$I = \int_0^1 \frac{2x + 3}{5x^2 + 1} \, dx$$

$$\int \frac{2x + 3}{5x^2 + 1} \, dx$$

$$= \frac{1}{5} \int \frac{5(2x + 3)}{5x^2 + 1} \, dx$$

$$= \frac{1}{5} \int \frac{(10x + 15)}{5x^2 + 1} \, dx$$

$$= \frac{1}{5} \int \frac{10x}{5x^2 + 1} \, dx + 3 \int \frac{1}{5x^2 + 1} \, dx$$

$$= \frac{1}{5} \int \frac{10x}{5x^2 + 1} \, dx + 3 \int \frac{1}{5\left(x^2 + \frac{1}{5}\right)} \, dx$$

$$= \frac{1}{5} \log (5x^2 + 1) + \frac{3}{5} \frac{1}{\frac{1}{\sqrt{5}}} \tan^{-1} \frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5} \log (5x^2 + 1) + \frac{3}{\sqrt{5}} \tan^{-1} (\sqrt{5})x$$

$$= F(x)$$

By second fundamental theorem of calculus, we get

$$I = F(3) - F(2)$$

$$= \frac{1}{5} \left[\log (1 + 5) - \frac{3}{\sqrt{5}} \log (\sqrt{5}) \right] - \left[\frac{1}{5} \log (1) + \frac{3}{\sqrt{5}} \tan^{-1} (0) \right]$$

$$= \frac{1}{5} [\log 6] + \frac{3}{\sqrt{5}} \tan^{-1} \sqrt{5}$$

15.

Evaluate the definite integral $\int_0^1 x^{x^2} \, dx$

Ans - To simplify the question let's say,

$$= \int_0^1 xe^{x^2} dx$$

$$x^2 = t$$

$$\Rightarrow 2x dx = dt$$

As $x \to 0, t \to 0$,

Again $x \to 1, t \to 1$,

$$\therefore I = \frac{1}{2} \int_0^1 e^t dt$$

$$\frac{1}{2} \int_0^1 e^t dt = \frac{1}{2} e^t dt$$

$$\frac{1}{2} e^t dt = F(t)$$

By second fundamental theorem of calculus, we get

$$I = F(1) - F(0)$$

$$= \frac{1}{2} e - \frac{1}{2} e^0$$

$$= \frac{1}{2}(e - 1)$$

16.

Evaluate the definite integral $\int_0^1 \frac{5x^2}{x^2+4x+3} dx$

Ans - To simplify the question let's say

$$I = \int_0^1 \frac{5x^2}{x^2 + 4x + 3} dx$$

Dividing $5x^2$ by $x^2 + 4x + 3$ we get

$$I = \int_1^2 \left\{ 5 - \frac{20x + 15}{x^2 + 4x + 3} \right\} dx$$

$$= \int_1^2 5\, dx - \int_1^2 \frac{20x + 15}{x^2 + 4x + 3} dx$$

$$= [5x]_1^2 - \int_1^2 \frac{20x + 15}{x^2 + 4x + 3} dx$$

$I = 5 - I_1$

Let, $20x + 15 = A\dfrac{d}{dx}(x^2 + 4x + 3) + B$

$= 2Ax + (4A + B)$

Equating coefficients of T and constant term, we get

$A = 10$ and $B = -25$

Let, $x^2 + 4x + 3 = t$

$\Rightarrow (2x + 4)dx = dt$

$\Rightarrow I_1 = 10\displaystyle\int \dfrac{dt}{t} - 25\displaystyle\int \dfrac{dx}{(x+2)^2 - 1^2}$

$= 10\log t - 25\left[\dfrac{1}{2}\log\left(\dfrac{x+1}{x+3}\right)\right]_1^2$

$= [10\log 15 - 10\log 18] - 25\left[\dfrac{1}{2}\log\dfrac{3}{5} - \dfrac{1}{2}\log\dfrac{2}{4}\right]$

$= [10\log 5 + 10\log 3 - 10\log 4 - 10\log 2]$

$\qquad\qquad - \dfrac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$

$$= \left[10 + \frac{25}{2}\right]\log 5 + \left[-10 - \frac{25}{2}\right]\log 4 + \left[10 + \frac{25}{2}\right]\log 3$$

$$+ \left[-10 + \frac{25}{2}\right]\log 2$$

$$= \frac{45}{2}\log 5 - \frac{45}{2}\log 4 - \frac{5}{2}\log 3 + \frac{5}{2}\log 2$$

$$= \frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}$$

Substituting the value of I_1 in (1), we get

$$I = 5 - \left[\frac{45}{2}\log\frac{5}{2} - \frac{5}{2}\log\frac{3}{2}\right]$$

$$= 5 - \frac{5}{2}\left[9\log\frac{5}{2} - \log\frac{3}{2}\right]$$

17.

Evaluate the definite integral $\displaystyle\int_0^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\,dx$

Ans - To simplify the question let's say

$$I = \int\limits_{0}^{\frac{\pi}{4}} (2\sec^2 x + x^3 + 2)\, dx$$

$$\int (2\sec^2 x + x^3 + 2)dx - 2\tan x + \frac{x^4}{4} + 2x$$

$$2\tan x + \frac{x^4}{4} + 2x = F(x)$$

By second fundamental theorem of calculus, we get

$$I = F\left(\frac{\pi}{4}\right) + F(0)$$

$$= \left\{ \left(2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^2 + 2\left(\frac{\pi}{4}\right) \right) - (2\tan 0 + 0 + 0) \right\}$$

$$= 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

18.

Evaluate the definite integral $\int_0^{\pi} \left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2} \right) dx$

Ans - To simplify the question let's say,

$$I = \int_0^\pi \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$$

$$= -\int_0^\pi \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right) dx$$

$$= -\int_0^\pi \cos x \, dx$$

$$= -\int_0^\pi \cos x \, dx = -\sin x$$

By second fundamental theorem of calculus, we get

$$I = F \left(\frac{\pi}{4} \right) + F(0)$$

$$= -\sin \pi + \sin 0 = 0$$

19.

Evaluate the definite integral $\int_0^1 \frac{6x+3}{x^2+4} dx$

Ans - To simplify the question let's say

$$I = \int_0^1 \frac{6x + 3}{x^2 + 4} \, dx$$

$$\int \frac{6x + 3}{x^2 + 4} \, dx$$

$$= 3 \int \frac{2x + 1}{x^2 + 4} \, dx$$

$$= 3 \int \frac{2x + 1}{x^2 + 4} \, dx$$

$$= 3 \int \frac{2x}{x^2 + 4} \, dx + 3 \int \frac{1}{x^2 + 4} \, dx$$

$$= 3 \log(x^2 + 4) + \frac{3}{2} \tan^{-1} \frac{x}{2}$$

$$= F(x)$$

By second fundamental theorem of calculus, we get

$$I = F(3) - F(2)$$

$$= 3 \left[\log(2^2 + 4) - \frac{3}{2} \tan^{-1} \left(\frac{2}{2} \right) \right]$$
$$\qquad - \left[3\log(0 + 4) + \frac{3}{2} \tan^{-1} \left(\frac{0}{2} \right) \right]$$

$$= 3[\log 8] + \frac{3}{2} \tan^{-1} 1 - 3\log 4 - \frac{3}{2} \tan^{-1} 0$$

$$= 3\log 8 + \frac{3}{2} \tan^{-1} 1 - 3\log 4 - \frac{3}{2} \tan^{-1} 0$$

$$= 3\log 8 + \frac{3}{2} \left(\frac{\pi}{4} \right) - 3\log 4 - 0$$

$$= 3\log \left(\frac{8}{4} \right) + \frac{3\pi}{8}$$

$$= 3\log 2 + \frac{3\pi}{8}$$

20.

Evaluate the definite integral $\int_0^1 \left(xe^x - \sin\dfrac{\pi x}{4} \right) dx$

Ans - To simplify the question let's say

$$I = \int_0^1 \left(xe^x - \sin\dfrac{\pi x}{4} \right) dx$$

$$= x\int e^x dx - \int \left\{ \left(\dfrac{d}{dx}x\right) \int e^x dx \right\} dx + \left\{ \dfrac{-\cos\dfrac{\pi x}{4}}{\dfrac{\pi}{4}} \right\} = F(x)$$

By second fundamental theorem of calculus we get

$$I = F(1) - F(0)$$

$$= \left(e^1 - e^1 - \dfrac{4}{\pi}\cos\pi\dfrac{1}{4} \right) - \left(0.\, e^0 - e^0 - \dfrac{4}{\pi}\cos 0 \right)$$

$$= e - e - \dfrac{4}{\pi}\left(\dfrac{1}{\sqrt{2}}\right) + 1 + \dfrac{4}{\pi}$$

$$= 1 + \dfrac{4}{\pi} - \dfrac{2\sqrt{2}}{\pi}$$

21.

Choose the correct answer $\displaystyle\int_1^{\sqrt{3}} \dfrac{dx}{1+x^2}$ equals

A. $\dfrac{\pi}{3}$ B. $\dfrac{2\pi}{3}$ C. $\dfrac{\pi}{6}$ D. $\dfrac{\pi}{12}$

Ans - To simplify the question let's say,

$$\int \frac{dx}{1+x^2} = \tan^{-1}x$$

By second fundamental theorem of calculus wo obtain

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = F(\sqrt{3}) - F(1)$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1}1$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Hence, correct answer is $\dfrac{\pi}{12}$

22.

Choose the correct answer $\int \dfrac{dx}{4+9x^2} = \int \dfrac{dx}{(2)^2+(3x)^2}$ **equals**

A. $\dfrac{\pi}{6}$ B. $\dfrac{\pi}{12}$ C. $\dfrac{\pi}{24}$ D. $\dfrac{\pi}{4}$

Ans - To simplify the question let's say,

$3x = t \Rightarrow 3dx = dt$

$$\int \frac{dx}{(2)^2 + (3x)^2} = \frac{1}{3} \int \frac{dt}{(2)^2 + (t)^2}$$

$$= \frac{1}{3} \left[\frac{1}{2} \tan^{-1} \frac{t}{2} \right]$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3x}{2} \right)$$

$$= F(x)$$

By second fundamental theorem of calculus,

$$\int_0^{\frac{2}{3}} \frac{dx}{4 + 9x^2} = F\left(\frac{2}{3} \right) - F(0)$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3}{2} \times \frac{2}{3} \right) - \frac{1}{6} \tan^{-1} 0$$

$$= \frac{1}{6} \tan^{-1} 1 - 0$$

$$= \frac{1}{6} \times \frac{\pi}{4}$$

$$= \frac{\pi}{24}$$

Hence, the correct answer is C. $\frac{\pi}{24}$

Exercise 7.9

1.

Evaluate the integral $\int_0^1 \frac{x}{x^2 + 1} dx$

Ans - To simplify the question let's take

$x^2 + 1 = t$

Taking derivative on both sides it will be,

$2x\, dx = dt$

When $x = 0$, then $t = 1$

When $x = 1$, then $t = 2$

$$\therefore \int_{0}^{1} \frac{x}{x^2 + 1}\, dx = \frac{1}{2} \int_{1}^{2} \frac{dt}{t}$$

$$= \frac{1}{2} [\log|t|]_{1}^{2}$$

$$= \frac{1}{2} [\log 2 - \log 1]$$

$$= \frac{1}{2} \log 2 (\because \log 1 = 0)$$

2.

Evaluate the integral $\int_{0}^{\frac{\pi}{2}} \sqrt{\sin\phi \cos^5\phi}\, d\phi$

Ans - To simplify the question, let's take

$\sin\phi = t$

$\Rightarrow \cos\phi\, d\phi = dt$

When, $\phi = 0, t = 0$

When $\phi = \dfrac{\pi}{2}, t = 1$

$$\therefore I = \int_0^1 \sqrt{t}(1-t^2)^2 dt$$

$$= \int_0^1 \sqrt{t}(1-t^2)^2 dt$$

$$= \int_0^1 \sqrt{t}(1-t^2)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}}(1+t^4-2t^2) dt$$

$$= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}} \right] dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} + \frac{2t^{\frac{7}{2}}}{\frac{7}{2}} \right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154 + 42 - 132}{231}$$

$$= \frac{64}{231}$$

3.

Evaluate the integral $\int_0^1 \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) dx$

Ans - To simplify the question, let's take

$x = \tan\theta$

Differentiating both side we get,

$dx = \sec^2\theta\, d\theta$

$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$

$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$

$= \int_0^{\frac{\pi}{4}} 2\theta \sec^2\theta\, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \theta \sec^2\theta\, d\theta$

Taking θ as first function and $\sec^2\theta$ as second function and integrating by parts, we get

$I = 2\left[\theta\int \sec^2\theta d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$

$= 2\left[\theta\tan\theta - \int \tan\theta d\theta\right]_0^{\frac{\pi}{4}}$

$= 2\left[\theta\tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$

$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$

$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$

$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$

$= \frac{\pi}{2} - \log 2$

4.

Evaluate the integral $\int_0^2 x\sqrt{x+2}\ \left(\text{Put}(x+2 = t^2)\right)$

Ans - To simplify the question let's take

$x + 2 = t^2$

Differentiating on both sides we get,

$dx = 2t\ dt$

When $x = 0, t = \sqrt{2}$

When $x = 2, t = 2$

$= \int_0^2 x\sqrt{x+2}\,dx$

$= \int_{\sqrt{2}}^2 (t^2 - 2)\sqrt{t^2}\,2dt$

$= 2\int_{\sqrt{2}}^2 (t^3 - 2t)dt$

$= 2\left[\dfrac{t^4}{4} - t^2\right]_{\sqrt{2}}^2$

$= 2[(4 - 4) - (1 - 2)] = 2$

5.

Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\,dx$

Ans - To simplify the question, let's take

$$\cos x = t$$

Differentiating on both sides we get,

$$-\sin x\, dx = dt$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\, dx = -\int_1^0 \frac{dt}{1 + t^2}$$

$$= -[\tan^{-1} 0 - \tan^{-1} 1]$$

$$= -\left[-\frac{\pi}{4}\right] \quad = \frac{\pi}{4}$$

6.

Evaluate the integral $\displaystyle\int_0^2 \frac{dx}{x + 4 - x^2}$

Ans - To simplify the question let's take

$$\int_0^2 \frac{dx}{x + 4 - x^2} = \int_0^2 \frac{dx}{-(x^2 - x - 4)}$$

Thus,

$$\int_0^2 \frac{dx}{x + 4 - x^2} = \int_0^2 \frac{dx}{-(x^2 - x - 4)}$$

$$= \int_0^2 \frac{dx}{-(x^2 - x + \frac{1}{4} - \frac{1}{4} - 4)}$$

$$= \int_0^2 \frac{dx}{-(x^2 - x + \frac{1}{4} - \frac{1}{4} - 4)}$$

$$= \int_0^2 \frac{dx}{-[(x - \frac{1}{2})^2 - \frac{17}{4}]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}$$

Let $x - \dfrac{1}{2} = t$

So, $dx = dt$

When $x = 0, t = -\dfrac{1}{2}$

When $x = 2, t = \dfrac{3}{2}$

$$= \int_0^2 \frac{dx}{2\left(\dfrac{\sqrt{17}}{2}\right) - \left(x - \dfrac{1}{2}\right)^2}$$

$$= \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\dfrac{\sqrt{17}}{2}\right)^2 - (t)^2}$$

$$= \left[\frac{1}{2\left(\dfrac{\sqrt{17}}{2}\right)} \log \frac{\dfrac{\sqrt{17}}{2} + t}{\dfrac{\sqrt{17}}{2} - t} \right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{17}} \left[\log \frac{\dfrac{\sqrt{17}}{2} + \dfrac{3}{2}}{\dfrac{\sqrt{17}}{2} - \dfrac{3}{2}} - \frac{\log \dfrac{\sqrt{17}}{2} - \dfrac{1}{2}}{\log \dfrac{\sqrt{17}}{2} + \dfrac{1}{2}} \right]$$

$$= \frac{1}{\sqrt{17}} \left[\log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} - \log \frac{\sqrt{17} - 1}{\sqrt{17} + 1} \right]$$

$$= \frac{1}{\sqrt{17}} \left[\log \frac{\sqrt{17} + 3}{\sqrt{17} - 3} \times \frac{\sqrt{17} - 1}{\sqrt{17} + 1} \right]$$

$$= \frac{1}{\sqrt{17}} \left[\log \frac{\sqrt{17} + 3 + 4\sqrt{17}}{\sqrt{17} + 3 - 4\sqrt{17}} \right]$$

$$= \frac{1}{\sqrt{17}} \left[\log \frac{20 + 4\sqrt{17}}{20 - 4\sqrt{17}} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{5 + \sqrt{17}}{5 - \sqrt{17}} \right)$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{(5 + \sqrt{17})(5 + \sqrt{17})}{25 - 17} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{(25 + 17 + 10\sqrt{17}}{8} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{(42 + 10\sqrt{17}}{8} \right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{(21 + 5\sqrt{17}}{4} \right]$$

7.

Evaluate the integral $\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$

Ans - To simplify the question let's take,

$$\int_{-1}^{1} \frac{dx}{x^2 + 2x + 1 + 4}$$

$$= \int_{-1}^{1} \frac{dx}{(x^2 + 2x + 1) + 4}$$

$$= \int_{-1}^{1} \frac{dx}{(x + 1)^2 + 2^2}$$

Let $x + 1 = t$

Differentiating both side we get,

$dx = dt$

When x = -1, t = 0

When x = 1, t = 2

$$\int_{-1}^{1} \frac{dx}{(x+1)^2 + 2^2}$$

$$= \int_{0}^{2} \frac{dx}{t^2 + 2^2}$$

$$= \left[\frac{1}{2} \tan^{-1} \frac{t}{2} \right]_{0}^{2}$$

$$= \frac{1}{2} \tan^{-1} 1 - \frac{1}{2} \tan^{-1} 0$$

$$= \frac{1}{2} \left(\frac{\pi}{4} \right)$$

$$= \frac{\pi}{8}$$

8.

Evaluate the integral $\int_{1}^{2} \left(\frac{1}{x} - \frac{1}{2x^2} \right) e^{2x} dx$

Ans - Let $2x = t$

After differentiating both side we get, $2dx = dt$

When $x = 1, t = 2$ and $x = 2, t = 4$

$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$

$= \frac{1}{2} \int_2^4 \left(\frac{2}{t^2} - \frac{2}{t^2}\right) e^t dt$

Let, $\frac{1}{t} = f(t)$

$f'(t) = -\frac{1}{t^2}$

$\int_2^4 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx = \frac{1}{2} \int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t dt$

$= \left[e^t \frac{1}{t}\right]_2^4$

$= \left[\frac{e^t}{t}\right]_2^4$

$= \frac{e^4}{t} - \frac{e^2}{t}$

$= \frac{e^2(e^2 - 2)}{4}$

9.

The value of the integral $\int_{\frac{1}{3}}^1 \frac{(x - x^3)^{\frac{1}{3}}}{x^4} dx$ **is,**

(A)6　　　　(B)0　　　　(C)3　　　　(D)4

Ans - Let $I = \int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4} dx$

Also, to simplify the question let's take,

$x = \sin \theta$

After differentiating both side we get

$dx = \cos \theta d\theta$

When, $x = \frac{1}{3}, \theta = \sin^{-1}\left(\frac{1}{3}\right)$

When $x = 1, \theta = \frac{\pi}{2}$

$$\Rightarrow I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{\sin \theta - \sin^3 \theta}{\sin^4 \theta} \cos \theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}}(1 - \sin^2 \theta)^{\frac{1}{3}}}{\sin^4 \theta} \cos \theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}}(\cos \theta)^{\frac{2}{3}}}{\sin^4 \theta} \cos \theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{2}\right)}^{\frac{\pi}{2}} \frac{(\sin \theta)^{\frac{1}{3}}(\cos \theta)^{\frac{2}{3}}}{\sin^2 \theta \sin^2 \theta} \cos \theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\cos \theta)^{\frac{5}{3}}}{(\sin \theta)^{\frac{5}{3}}} \csc^2 \theta \, d\theta$$

$$= \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} (\cot \theta)^{\frac{5}{3}} \csc 2\theta \, d\theta$$

Let $\cot \theta = t$

Differentiating on both side we get,

$-\operatorname{cosecc}^2 \theta \, d\theta = dt$

When $\theta = \sin^{-1}\left(\frac{1}{3}\right), t = 2\sqrt{2}$

When $\theta = \frac{\pi}{2}, t = 0$

$\therefore I = -\int_{2\sqrt{2}}^{0} (t)^{\frac{5}{3}} dt$

$= -\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$

$= -\frac{3}{8}\left[-(2\sqrt{2})^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$

$= \frac{3}{8}\left[(\sqrt{8})^{\frac{8}{3}}\right]$

$= \frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$

$= \frac{3}{8}[16]$

$= 3 \times 2 = 6$

Hence, the correct Answer is option A.

10.

If $f(x) = \int_0^x t \sin t \, dt$, then $f'(x)$ is

(A) $\cos x + x\sin x$ (B) $x\sin x$

(C) $x\cos x$ (D) $\sin x + x\cos x$

Ans - $f(x) = \int_0^x t\sin t \, dt$

Integrating by parts, we get

$$f(x) = t \int_0^x \sin t \, dt - \int_0^x \left\{ \left(\frac{d}{dx} t \right) \int \sin t \, dt \right\} dt$$

$$= [t(-\cos t)]_0^x - \int_0^x (-\cot t) \, dt$$

$$= [-t\cos t + \sin x]_0^x$$

$$= [-x\cos x + \sin t]_0^x$$

$$= -x\cos x + \sin x$$

$$\Rightarrow f'(x) = -[\{x(-\sin x)\} + \cos x] + \cos x$$

$$= x\sin x - \cos x + \cos x$$

$$= x\sin x$$

Hence, the correct answer is option B.

Exercise 7.10

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

1.

$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

Ans - Given $I = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ ..... (1)

We know that,

$\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Therefore, the integral becomes

$I = \int_0^{\frac{\pi}{2}} \cos^2 \left(\frac{\pi}{2} - x\right) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

Adding equation (1) and (2),

$2I = \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\,dx$

$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$

$\Rightarrow 2I = \dfrac{\pi}{2}$

$\Rightarrow I = \dfrac{\pi^2}{4}$

$\therefore \dfrac{\pi}{2}$

$\therefore \int_0^2 \cos^2 x\,dx = \dfrac{\pi}{4}$

2.

$\int_0^{\frac{\pi}{2}} \dfrac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}dx$

Ans - Given $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ ... ... (1)

We know that,

$$\int_0^a f(x)dx = \int_0^a f(a - x)dx$$

Therefore, the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \text{... ... (2)}$$

Adding equation (1) and (2),

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \frac{\pi}{4}$$

3.

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x\, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$$

Ans - Given

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin \frac{3}{2}x \, dx}{\sin \frac{3}{2}x + \cos \frac{3}{2}x} \qquad \dots (1)$$

We know that,

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

Therefore, the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} \left(\frac{\pi}{2} - x\right) dx}{\sin^{\frac{3}{2}} \left(\frac{\pi}{2} - x\right) + \sin^{\frac{3}{2}} \left(\frac{\pi}{2} - x\right)}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}}x \, dx}{\sin^{\frac{3}{2}}x + \sin^{\frac{3}{2}}x} \qquad \dots (2)$$

Adding equation (1) and (2),

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}x + \cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x + \sin^{\frac{3}{2}}x} \, dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x \, dx}{\sin \frac{3}{2}x + \cos \frac{2}{2}x} = \frac{\pi}{4}$$

4.

$$\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$$

Ans - Given

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x}$$

We know that,

$$\int_0^a f(x) dx = \int_0^a f(a - x) dx$$

Therefore, the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 \left(\frac{\pi}{2} - x\right) dx}{\sin^5 \left(\frac{\pi}{2} - x\right) + \cos^5 \left(\frac{\pi}{2} - x\right)}$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x dx}{\sin^5 x + \cos^5 x} \qquad \dots (2)$$

Adding equation (1) and (2),

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\sin^5 x + \cos^5 x}{\sin^5 x + \cos^5 x} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1 dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi^2}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x + \cos^5 x} = \frac{\pi}{4}$$

5.

$\int_{-5}^{5} |x + 2| dx$

Ans - Let I = $\int_{-5}^{5} |x + 2| dx$

Since, $(x + 2) \le 0$ for interval $[-5, -2]$.

Therefore, $(x + 2) \ge 0$ for interval $[-2, 5]$

As, $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

Hence, $\int_{-5}^{-2} - (x + 2) dx + \int_{-2}^{5} (x + 2) dx$.

$\Rightarrow I = \int_{-5}^{-2} - (x + 2) dx + \int_{-2}^{5} (x + 2) dx$

$= -\left[\dfrac{x^2}{2} + 2x\right]_{-5}^{2} + \left[\dfrac{x^2}{2} + 2x\right]_{-2}^{5}$

$= -\left[\dfrac{(2)^2}{2} + 2(2) - \dfrac{(-5)^2}{2} - 2(-5)\right]$

$\qquad + \left[\dfrac{(5)^2}{2} + 2(5) - \dfrac{(-2)^2}{2} - 2(-2)\right]$

$= -\left[2 - 4 - \dfrac{25}{2} + 10\right] + \left[\dfrac{25}{2} + 10 - 2 + 4\right]$

$= -2 + 4 + \dfrac{25}{2} + 10 + \dfrac{25}{2} + 10 - 2 + 4$

$= 29$

6.

$\int_{2}^{8} |x - 5| dx$

Ans - Let $I = \int_{2}^{8} |x - 5| dx$

Since, $(x - 5) \leq 0$ for interval $[2,5]$

Therefore, $(x - 5) \geq 0$ for interval $[5,8]$

As, $\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$

Hence, $\int_{2}^{5} - (x - 5)dx + \int_{5}^{8} (x - 5)dx$

$\Rightarrow I = \int_{2}^{5} - (x - 5)dx + \int_{5}^{8} (x - 5)dx$

$$= -\left[\frac{x^2}{2} - 5x\right]_{2}^{5} + \left[\frac{x^2}{2} - 5x\right]_{5}^{8}$$

$$= -\left[\frac{(5)^2}{2} - 5(5) - \frac{(2)^2}{2} + 5(2)\right]$$

$$+ \left[\frac{(8)^2}{2} - 5(8) - \frac{(5)^2}{2} + 5(5)\right]$$

$$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$$

$$= -\frac{25}{2} + 25 + 2 - 10 + 32 - 40 - \frac{25}{2} + 25$$

$$= 9$$

7.

$\int_{0}^{1} x(1 - x)^{n} dx$

Ans - Let $I = \int_0^1 x(1-x)^n dx$

$\Rightarrow I = \int_0^1 (1-x)(1-(1-x))^n dx$

Since, $\int_1^a f(x)dx = \int_0^a f(a-x)dx$.

$\Rightarrow \int_0^1 (1-x)(x)^n dx$

$= \int_0^1 (x^n - x^{n+1})dx$

$= \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^1$

$= \left[\frac{1}{n+1} - \frac{1}{n+2} \right]^0$

$= \frac{(n+2)-(n+1)}{(n+1)(n+2)}$

$= \frac{1}{(n+1)(n+2)}$

8.

$\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$

Ans - Let $I = \int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$

Since, $\int_0^a f(x) dx = \int_0^a f(a - x) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \log \left[1 + \tan \left(\frac{\pi}{4} - x \right) \right] dx$

$= \int_0^{\frac{\pi}{4}} \log \left[1 + \dfrac{\tan \frac{\pi}{4} - \tan x}{1 + \tan \frac{\pi}{4} \tan x} \right] dx$

$= \int_0^{\frac{\pi}{4}} \log \left\{ 1 + \dfrac{1 - \tan x}{1 + \tan x} \right\} dx$

$= \int_0^{\frac{\pi}{4}} \log \dfrac{2}{1 + \tan x} dx$

$= \int_0^{\frac{\pi}{4}} \log 2 dx - \int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$

$= \int_0^{\frac{\pi}{4}} \log 2 dx - I$

$2I = [x \log 2]_0^{\frac{\pi}{4}}$

$2I = \dfrac{\pi}{4} \log 2$

$I = \dfrac{\pi}{8} \log 2$

9.

$\int_0^2 x \sqrt{2 - x} \, dx$

Ans - Let $I = \int_0^2 x\sqrt{2-x}\,dx$

Since, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\Rightarrow I = \int_0^2 (2-x)\sqrt{x}\,dx$

$= \int_0^2 \{2x^{1/2} - x^{3/2}\}dx$

$= \left[2\left(\dfrac{x^{3/2}}{3/2}\right) - \dfrac{x^{5/2}}{5/2}\right]_0^2$

$= \left[\dfrac{4}{3}x^{3/2} - \dfrac{2}{5}x^{5/2}\right]_0^2$

$= \dfrac{4}{3}(2)^{3/2} - \dfrac{2}{5}(2)^{5/2}$

$= \dfrac{8\sqrt{2}}{3} - \dfrac{8\sqrt{2}}{5}$

$= \dfrac{40\sqrt{2} - 24\sqrt{2}}{15}$

$= \dfrac{16\sqrt{2}}{15}$

10.

$\int_0^{\frac{\pi}{2}}(2\log\,\sin\,x - \log\,\sin\,2x)dx$

Ans - Let $I = \int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin 2x)dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} (2\log \sin x - \log (2\sin x\cos x))dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} (2\log \sin x - \log \sin x - \log \cos x - \log 2)dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log \sin x - \log \cos x - \log 2)dx$ ..... (1)

Since, $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log \cos x - \log \sin x - \log 2)dx$ ... (2)

On adding equation (1) and (2)

$2I = \int_0^{\frac{\pi}{2}} (-\log 2 - \log 2)dx$

$\Rightarrow 2I = -2\log 2 \int_0^{\frac{\pi}{2}} dx$

$\Rightarrow I = -\log 2 \left[\frac{\pi}{2}\right]$

$\Rightarrow I = -\frac{\pi}{2} [\log 2]$

$\Rightarrow I = \frac{\pi}{2} [-\log 2]$

$\Rightarrow I = \frac{\pi}{2} \left[\log \frac{1}{2}\right]$

$\Rightarrow I = \frac{\pi}{2} \log \frac{1}{2}$

11.

$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

Ans - Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx$

Since, $\sin^2 x$ is an even function.

Therefore, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx = 2\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$

As if $f(x)$ is an even function, then

$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$

$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$

$= 2\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} dx$

$= \int_0^{\frac{\pi}{2}} (1 - \cos 2x)dx$

$= \left[x - \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$

12.

$\int_0^{\pi} \frac{x}{1 + \sin x} dx$

Ans – Let I $= \int_0^\pi \dfrac{x}{1+\sin\ x}dx$ ...... (1)

Since, $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$

Therefore, I $= \int_0^\pi \dfrac{(\pi-x)}{1+\sin\ x(\pi-x)}dx$

$I = \int_0^\pi \dfrac{(\pi-x)}{1+\sin\ x}dx$ .. (2)

On adding equation (1) and (2)

$2I = \int_0^\pi \dfrac{\pi}{1+\sin\ x}dx$

$\Rightarrow 2I = \pi\int_0^\pi \dfrac{(1-\sin\ x)}{(1+\sin\ x)+(1-\sin\ x)}dx$

$\Rightarrow 2I = \pi\int_0^\pi \dfrac{(1-\sin\ x)}{\cos^2\ x}dx$

$\Rightarrow 2I = \pi\int_0^\pi \{\sec^2\ x - \tan\ x\sec\ x\}dx$

$\Rightarrow 2I = \pi[2]$

$\Rightarrow I = \pi$

13.

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7\ xdx$$

Ans - Let I $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x \, dx$

Since, $\sin^7 x$ is an even function.

$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \, dx = 0$

As if $f(x)$ is an odd function, then $\int_{-a}^{a} f(x) \, dx = 0$.

Hence, $I = 0$

14.

$\int_{0}^{2\pi} \cos^5 x \, dx$

Ans – Let I $= \int_{0}^{2\pi} \cos^5 x \, dx$

$\cos^5(2\pi - x) = \cos^5 x \, \dots (1)$

If $f(2a - x) = f(x)$ then,

$\int_{0}^{2a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx$

If $f(2a - x) = -f(x)$ then,

$\int_{0}^{2a} f(x) \, dx = 0$

Since, $\cos^5(\pi - x) = -\cos^5 x$

$\Rightarrow I = 2 \int_{0}^{2\pi} \cos^5 x \, dx$

$\Rightarrow I = 2(0)$

$\Rightarrow I = 0$

15.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$$

Ans - Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ ...... (1)

Since, $\int_0^a f(x) dx = \int_0^a f(a - x) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \dfrac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \dfrac{\cos x - \sin x}{1 + \cos x \sin x} dx$ ...... (2)

On adding equation (1) and (2)

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \dfrac{0}{1 + \cos x \sin x} dx$

$\Rightarrow I = 0$

16.

$$\int_0^{\pi} \log (1 + \cos x) dx$$

Ans - Let

$$I = \int_0^{\pi} \log(1 + \cos x) \, dx \ ...... (1)$$

Since, $\int_0^a f(x) dx = \int_0^a f(a - x) dx$

$\Rightarrow I = \int_0^{\pi} \log (1 + \cos (\pi - x)) dx$

$\Rightarrow I = \int_0^{\pi} \log(1 - \cos x) \, dx \ ... (2)$

On adding equation (1) and (2)

$$2I = \int_0^\pi \{\log(1 - \cos x) + \log(1 - \cos x)\}dx$$

$$\Rightarrow 2I = \int_0^\pi \log(1 - \cos^2 x)dx$$

$$\Rightarrow 2I = \int_0^\pi \log \sin^2 x\,dx$$

$$\Rightarrow 2I = 2\int_0^\pi \log \sin x\,dx$$

$$\Rightarrow I = \int_0^\pi \log \sin x\,dx \dots (3)$$

Since, $\sin(\pi - x) = \sin x$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log \sin x\,dx \quad \dots (4)$$

$$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right)dx$$

$$\Rightarrow I = 2\int_0^{\frac{\pi}{2}} \log \cos x\,dx \quad \dots (5)$$

On adding equation (4) and (5)

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x + \log 2 - \log 2) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log 2\sin x\cos x - \log 2) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} (\log 2\sin x\cos x) dx - \int_0^{\frac{\pi}{2}} \log 2 dx$$

Let $2x = t$

On differentiating,

$2dx = dt$

If $x = 0$ then $t = 0$

$$\Rightarrow I = \frac{I}{2} - \frac{\pi}{2}\log 2$$

$$\Rightarrow \frac{I}{2} = -\frac{\pi}{2}\log 2$$

$$\Rightarrow I = -\pi\log 2$$

17.

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx$$

Ans - Let

$$I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}} dx \quad \dots (1)$$

Since, $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

$$I = \int_0^a \frac{\sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} dx \quad \dots (2)$$

On adding equation (1) and (2)

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a - x}}{\sqrt{a - x} + \sqrt{x}} dx$$

$$\Rightarrow 2I = \int_0^a dx$$

$$\Rightarrow 2I = [x]_0^a$$

$$\Rightarrow 2I = a$$

$$\Rightarrow I = \frac{a}{2}$$

18.

$$\int_0^4 |x - 1| dx$$

Ans - Let $I = \int_0^4 |x - 1| dx$

$(x - 1) \leq 0$ when $0 \leq x \leq 1$

$(x - 1) \geq 0$ when $1 \leq x \leq 4$

Since, $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x)$

$\Rightarrow I = \int_0^1 |x - 1| dx + \int_1^4 |x - 1| dx$

$\Rightarrow I = \int_0^1 - (x - 1) dx + \int_1^4 (x - 1) dx$

$= \left[x - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^4$

$= 1 - \frac{1}{2} + \frac{(4)^2}{2} - 4 - \frac{1}{2} + 1$

$= 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1$

$= 5$

19.

Show that $\int_0^a f(x) g(x) dx = 2 \int_0^a f(x) dx$, if f and g are defined as $f(x) = f(a - x)$ and $g(x) + g(a - x) = 4$.

Ans - Let $\int_0^a f(x)g(x)dx$ ... (1)

Since, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\Rightarrow \int_0^a f(a-x)g(a-x)dx$

$\Rightarrow \int_0^a f(x)g(a-x)dx$ ... (2)

On adding equation (1) and (2)

$2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\}dx$

$\Rightarrow 2I = \int_0^a f(x)\{g(x) + g(a-x)\}dx$

As, $g(x) + g(a-x) = 4$.

$\Rightarrow 2I = \int_0^a 4f(x)dx$

$\Rightarrow I = 2\int_0^a f(x)dx$

Hence, $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$, if f and g are defined as
$f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$.

20.

The value of $\int_{-\pi}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)dx$ is

(A)0   (B)2   (C)π   (D)1

Ans - Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x^3 + x\cos x + \tan^5 x + 1)dx$

$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x^3)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x\cos x)dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\tan^5 x)dx$

$\qquad\qquad + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1)dx$

If $f(x)$ is an even function, then $\int_{-a}^{a}f(x)dx = 2\int_{0}^{a}f(x)dx$

If $f(x)$ is an odd function, then $\int_{-a}^{a}f(x)dx = 0$

$\Rightarrow I = 0 + 0 + 0 + 2\int_{0}^{\frac{\pi}{2}}dx$

$= 2[x]_{0}^{\frac{\pi}{2}}$

$= 2\left[\frac{\pi}{2}\right] = \pi$

Hence, correct option is C.

21.

The value of $\int_{0}^{\frac{\pi}{2}}\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ **is**

(A)2   (B)$\frac{3}{4}$ (C)0   (D)−2

Ans - Let $I = \int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ ... (1)

Since, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\frac{4 + 3\sin\left(\frac{\pi}{2} - x\right)}{4 + 3\cos\left(\frac{\pi}{2} - x\right)}\right) dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left(\frac{4 + 3\cos x}{4 + 3\sin x}\right) dx \quad ... (2)$$

On adding equation (1) and (2)

$2I = \int_0^{\frac{\pi}{2}} \log 1 \, dx$

$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0 \, dx$

$\Rightarrow I = 0$

Hence, correct option is C.

Miscellaneous Exercise

Integrate the functions in Exercises 1 to 23.

1.

$$\frac{1}{x - x^3}$$

Ans - Given $\dfrac{1}{x-x^3}$

$$\Rightarrow \frac{1}{x-x^3} = \frac{1}{x(1-x^2)}$$

$$= \frac{1}{x(1-x)(1-x)}$$

Let $\dfrac{1}{x(1-x)(1+x)} = \dfrac{A}{x} + \dfrac{B}{(1-x)} + \dfrac{C}{1+x}$ $\qquad$ ... (1)

$$\Rightarrow 1 = A(1-x^2) + Bx(1+x) + Cx(1-x)$$

$$\Rightarrow 1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$$

On equating coefficients of x^2, x and constant term

$$-A + B - C = 0$$

$$B + C = 0$$

$$A = 1$$

$$\Rightarrow A = 1, B = \frac{1}{2} \text{ and } C = -\frac{1}{2}$$

From equation (1)

$$\frac{1}{x(1-x)(1+x)} = \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$

$$\Rightarrow \int \frac{1}{x(1-x)(1+x)} dx$$

$$= \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{(1-x)} dx - \frac{1}{2} \int \frac{1}{(1+x)} dx$$

$$= \log x - \frac{1}{2} \log(1-x) - \frac{1}{2} \log(1+x)$$

$$= \log x - \frac{1}{2} \log(1-x)^{\frac{1}{2}} - \frac{1}{2} \log(1+x)^{\frac{1}{2}}$$

$$= \log \left(\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}} \right) + C$$

$$= \log \left(\frac{x^2}{(1-x^2)} \right)^{\frac{1}{2}} + C$$

$$= \frac{1}{2} \log \left(\frac{x^2}{1-x^2} \right) + C$$

2.

$$\frac{1}{\sqrt{x+a} + \sqrt{x+b}}$$

Ans - Given $\frac{1}{\sqrt{x+a}+\sqrt{x+b}}$

$$= \frac{1}{\sqrt{x+a} + \sqrt{x+b}} \times \frac{\sqrt{x+a} - \sqrt{x+b}}{\sqrt{x+a} - \sqrt{x+b}}$$

$$= \frac{\sqrt{x+a} - \sqrt{x+b}}{(x+a) - (x+b)}$$

$$= \frac{\sqrt{x+a} - \sqrt{x+b}}{a - b}$$

$$\Rightarrow \int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx = \frac{1}{a-b} \int (\sqrt{x+a} - \sqrt{x+b}) dx$$

$$= \frac{1}{a-b} \left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}} \right]$$

$$= \frac{2}{3(a-b)} \left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right] + C$$

3.

$$\frac{1}{x\sqrt{ax - x^2}}$$

Ans - Given $\dfrac{1}{x\sqrt{ax-x^2}}$

Let $x = \dfrac{a}{t}$

On differentiating we get,

$dx = -\dfrac{a}{t^2}dt$

$\Rightarrow \displaystyle\int \dfrac{1}{x\sqrt{ax-x^2}}dx = \int \dfrac{1}{\dfrac{a}{t}\sqrt{a\cdot\dfrac{a}{t}-\left(\dfrac{a}{t}\right)^2}}\left(-\dfrac{a}{t^2}dt\right)$

$= -\displaystyle\int \dfrac{1}{at}\cdot\dfrac{1}{\sqrt{\dfrac{1}{t}-\dfrac{1}{t^2}}}dt$

$= -\dfrac{1}{a}\displaystyle\int \dfrac{1}{\sqrt{\dfrac{t^2}{t}-\dfrac{t^2}{t^2}}}dt$

$= -\dfrac{1}{a}\displaystyle\int \dfrac{1}{\sqrt{t-1}}dt$

$= -\dfrac{1}{a}\left[2\sqrt{t-1}\right]+C$

Substituting value of t we get

$= -\dfrac{1}{a}\left[2\sqrt{\dfrac{a}{x}-1}\right]+C$

$= -\dfrac{2}{a}\left[\dfrac{\sqrt{a-x}}{\sqrt{x}}\right]+C$

$= -\dfrac{2}{a}\left[\sqrt{\dfrac{a-x}{x}}\right]+C$

4.

$$\dfrac{1}{x^2(x^4+1)^{\frac{3}{4}}}$$

Ans - Given expression is, $\dfrac{1}{x^2(x^4+1)^{\frac{3}{4}}}$

On multiplying and dividing by x^{-3} we get

$$\dfrac{x^{-3}}{x^2 . x^{-3}(x^4+1)^{\frac{3}{4}}} = \dfrac{x^{-3}(x^4+1)^{\frac{-3}{4}}}{x^2 - x^{-3}} = \dfrac{(x^4+1)^{\frac{-3}{4}}}{x^5(x^4)^{\frac{-3}{4}}}$$

$$= \dfrac{1}{x^5}\left(\dfrac{x^4+1}{x^4}\right)^{\frac{-3}{4}}$$

$$= \dfrac{1}{x^5}\left(1+\dfrac{1}{x^4}\right)^{\frac{-3}{4}}$$

Let $\dfrac{1}{x^4} = t$

$$\therefore \dfrac{-4}{x^5} = \dfrac{dt}{dx}$$

$$\Rightarrow \dfrac{dx}{x^5} = \dfrac{-dt}{4}$$

$$\therefore \int \dfrac{1}{x^2(x^4+1)^{\frac{3}{4}}}dx = \int \dfrac{1}{x^5}\left(1+\dfrac{1}{x^4}\right)^{\frac{-3}{4}}dx = \dfrac{-1}{4}\int (1+t)^{\frac{-3}{4}}dt$$

$$\Rightarrow \int \dfrac{1}{x^2(x^4+1)^{\frac{3}{4}}}dx = \dfrac{-1}{4}\left[\dfrac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right] + C = -\left(1+\dfrac{1}{x^4}\right)^{\frac{1}{4}} + C,$$

where C is any arbitrary constant.

5.

$$\dfrac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} . \left[\text{Hint: } \dfrac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \dfrac{1}{x^{\frac{1}{3}}\left(1 + x^{\frac{1}{6}}\right)} \text{ Put, } x = t^6\right]$$

Ans - Given expression is, $\dfrac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$.

From given hint we get,

$$\therefore \dfrac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}} = \dfrac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}$$

Consider $x = t^6$

$$\therefore x = t^6 \Rightarrow dx = 6t^5$$

$$\therefore \int \dfrac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx = \int \dfrac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)}dx = \int \dfrac{6t^5}{t^2(1+t)}dt$$

$$= 6\int \dfrac{t^3}{(1+t)}dt$$

Now on dividing, we can get as shown below,

$$\int \dfrac{1}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}dx = 6\int \left\{(t^2-t+1) - \dfrac{1}{1+t}\right\}dt$$

$$= 6\int \left[\left(\dfrac{t^3}{3}\right) - \left(\dfrac{t^2}{2}\right) + t - \log|1+t|\right]dt$$

$$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1+x^{\frac{1}{6}}\right) + C$$

$$= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1+x^{\frac{1}{6}}\right) + C$$

where C is any arbitrary constant.

6.

$$\dfrac{5x}{(x+1)(x^2+9)}$$

Ans - Given expression is, $\dfrac{5x}{(x+1)(x^2+9)}$

Now consider it as shown below

$$\therefore \frac{5x}{(x+1)(x^2+9)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+9)} \quad ...(1)$$

$$\Rightarrow 5x = A(x^2+9) + Bx + C(x+1)$$

$$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$$

On equating coefficients of x^2, x and constant term, it can be obtained that,

$$A + B = 0 \;...(2)$$

$$B + C = 5 \;...(3)$$

$$9A + C = 0 \;...(4)$$

And on solving these equations, values of A, B, C will be

$$A = \frac{-1}{2}, B = \frac{1}{2}, C = \frac{9}{2}$$

Now, from equation (1) it can be clearly obtained that,

$$\int \frac{5x}{(x+1)(x^2+9)}dx = \int \left\{\frac{-1}{2(x+1)} + \frac{x+9}{2(x^2+9)}\right\}dx$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{2}\int \frac{x}{x^2+9}dx + \frac{9}{2}\int \frac{1}{x^2+9}dx$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \left(\frac{9}{2}\right)\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{x}{3}\right)$$

$$= \frac{-1}{2}\log|x+1| + \frac{1}{4}\log|x^2+9| + \frac{3}{2}\tan^{-1}\left(\frac{x}{3}\right) + C$$

where C is any arbitrary constant.

7.

$$\frac{\sin x}{\sin(x-\alpha)}$$

Ans - Given expression is, $\dfrac{\sin x}{\sin(x - \alpha)}$

Substitute $x - \alpha = t$

$\therefore dx = dt$

$\therefore \int \dfrac{\sin x}{\sin(x - \alpha)} dx = \int \dfrac{\sin(t + \alpha)}{\sin t} dt$

$$= \int \dfrac{\sin t \cos\alpha + \cos t \sin\alpha}{\sin t} dt$$

$$= \int \cos\alpha + \cot t \sin\alpha\, dt$$

$\Rightarrow \int \dfrac{\sin x}{\sin(x - \alpha)} dx = t\cos\alpha + \sin\alpha \log|\sin t| + C_1$

$$= (x - \alpha)\cos\alpha + \sin\alpha \log|\sin(x - \alpha)| + C_1$$

$\Rightarrow \int \dfrac{\sin x}{\sin(x - \alpha)} dx$

$$= x\cos\alpha + \sin\alpha\log|\sin(x - \alpha)| - \alpha\cos\alpha + C_1$$
$$= x\cos\alpha + \sin\alpha\log|\sin(x - \alpha)| + C$$

where C_1, C are any arbitrary constants and $C = C_1 - \alpha\cos\alpha$

8.

$$\dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$$

Ans - Given expression is, $\dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

$\therefore \dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \dfrac{e^{4\log x}(e^{\log x} - 1)}{e^{2\log x}(e^{\log x} - 1)} = e^{2\log x} = x^2$

Now, integrate the given expression as shown below

$\therefore \int \dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} dx = \int x^2 dx = \dfrac{x^3}{3} + C$

where C is any arbitrary constant.

9.

$$\dfrac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Ans - Given expression is, $\dfrac{\cos x}{\sqrt{4-\sin^2 x}}$

Substitute $\sin x = t$

Differentiating on both sides we get,

$\cos x\, dx = dt$

$\therefore \int \dfrac{\cos x}{\sqrt{4-\sin^2 x}} dx = \int \dfrac{dt}{\sqrt{(2)^2-(t)^2}} = \sin^{-1}\left(\dfrac{t}{2}\right) + C$

$\qquad\qquad = \sin^{-1}\left(\dfrac{\sin x}{2}\right) + C$

C is any arbitrary constant.

10.

$$\dfrac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x}$$

Ans - Given expression is, $\dfrac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x}$

$\therefore \dfrac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x}$

$\qquad = \dfrac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{\sin^2 x + \cos^2 x - \sin^2 x\cos^2 x - \sin^2 x\cos^2 x}$

$= \dfrac{(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)(\sin^4 x + \cos^4 x)}{\sin^2 x(1 - \cos^2 x) + \cos^2 x(1 - \sin^2 x)}$

$= \dfrac{-(-\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)}{\sin^4 x + \cos^4 x} = -\cos 2x$

Integrate given expression we get

$\therefore \int \dfrac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x} dx$

$= \int -\cos 2x\, dx = \dfrac{-\sin 2x}{2} = +C$

where C is any arbitrary constant.

11.

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Ans - Given expression is $\frac{1}{\cos(x+a)\cos(x+b)}$.

On multiplying and dividing by $\sin(\alpha - \beta)$ we get,

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$\Rightarrow \frac{1}{\sin(a-b)}\left[\frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)}\right]$$

$$\Rightarrow \frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$\Rightarrow \frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$= \frac{1}{\sin(a-b)}(\tan(x+a) - \tan(x+b))$$

$$\therefore \int \frac{1}{\cos(x+a)\cos(x+b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\int (\tan(x+a) - \tan(x+b))\,dx$$

$$\Rightarrow \int \frac{1}{\cos(x+a)\cos(x+b)}\,dx$$

$$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x+b)}{\cos(x-a)}\right|\right] + C$$

where C is any arbitrary constant.

12.

$$\frac{x^3}{\sqrt{1-x^8}}$$

Ans - Given expression is, $\dfrac{x^3}{\sqrt{1-x^8}}$

Now, substitute $x^4 = t$

Differentiating on both sides we get,

$\therefore 4x^3 dx = dt$

$\therefore \int \dfrac{x^3}{\sqrt{1-x^8}} dx = \dfrac{1}{4} \int \dfrac{dt}{\sqrt{1-(t)^2}} = \dfrac{1}{4} \sin^{-1} t + C$

$= \dfrac{1}{4} \sin^{-1}(x^4) + C$

where C is any arbitrary constant.

13.

$$\dfrac{e^x}{(1+e^x)(2+e^x)}$$

Ans - Given expression is, $\dfrac{e^x}{(1+e^x)(2+e^x)}$.

Now, substitute $e^x = t$

Differentiating on both sides we get,

$\therefore e^x dx = dt$

$\therefore \int \dfrac{e^x}{(1+e^x)(2+e^x)} dx = \int \dfrac{dt}{(t+1)(t+2)}$

$= \int \left[\dfrac{1}{(t+1)} - \dfrac{1}{(t+2)} \right] dt$

$\int \dfrac{e^x}{(1+e^x)(2+e^x)} dx = \log|t+1| - \log|t+2| + C$

$= \log \left| \dfrac{e^x+1}{e^x+2} \right| + C$

where C is any arbitrary constant

14.

$$\dfrac{1}{(x^2+1)(x^2+4)}$$

Ans - Given expression is, $\dfrac{1}{(x^2 + 1)(x^2 + 4)}$

Now consider it as shown below

$$\therefore \dfrac{1}{(x^2 + 1)(x^2 + 4)} = \dfrac{Ax + B}{(x^2 + 1)} + \dfrac{Cx + D}{(x^2 + 4)} \quad \dots (1)$$

$$\Rightarrow 1 = (Ax + B)(x^2 + 4) + (Bx + C)(x^2 + 9)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

On equating coefficients of x^3, x^2, x and constant term, it can be obtained that

$A + C = 0 \dots (2)$

$B + D = 0 \dots (3)$

$4A + C = 0 \dots (4)$

$4B + D = 1 \dots (5)$

On solving these equations, the values of A B, C, D will be,

$$A = 0, B = \dfrac{1}{3}, C = 0, D = \dfrac{1}{3}$$

$$\int \dfrac{1}{(x^2 + 1)(x^2 + 4)}\, dx = \dfrac{1}{3}\int \left\{ \dfrac{1}{(x^2 + 1)} - \dfrac{1}{(x^2 + 4)} \right\} dx$$

$$= \dfrac{1}{3}\tan^{-1}x - \dfrac{1}{(3)(2)}\tan^{-1}\dfrac{x}{2} + C$$

$$= \dfrac{1}{3}\tan^{-1}x - \dfrac{1}{6}\tan^{-1}\dfrac{x}{2} + C$$

where C is any arbitrary constant.

15.

$\cos^3 x e^{\log \sin x}$

Ans - Given expression is, $\cos^3 x e^{\log \sin x}$

$\therefore \cos^3 x e^{\log \sin x} = \cos^3 x \sin x$

Now, consider $\cos x = t$

$\therefore -\sin x \, dx = dt$

$\therefore \int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x \sin x dx = -\int t^3 dt = \dfrac{-t^4}{4} + C$

$$= \dfrac{-\cos^4 x}{4} + C$$

where C is any arbitrary constant.

16.

$$e^{3 \log x}(x^4 + 1)^{-1}$$

Ans - Given expression is, $e^{3 \log x}(x^4 + 1)^{-1}$

$\therefore e^{3 \log x}(x^4 + 1)^{-1} = \dfrac{x^3}{(x^4 + 1)}$

Now, consider $x^4 + 1 = t$

$\Rightarrow 4x^3 dx = dt$

$\therefore \int e^{3 \log x}(x^4 + 1)^{-1} dx = \int \dfrac{x^3}{x^4 + 1} dx = \dfrac{1}{4} \int \dfrac{dt}{t}$

$$= \dfrac{1}{4} \log|t| + C = \dfrac{\log|x^4 + 1|}{4} + C$$

where C is any arbitrary constant.

17.

$$f'(ax + b)[f(ax + b)]^n$$

Ans - Given expression is, $f'(ax+b)[f(ax+b)]^n$

Let $[f(ax+b)] = t$

Differentiating on both sides we get,

$\Rightarrow af'(ax+b)dx = dt$

$\therefore \int f'(ax+b)[f(ax+b)]^n dx = \frac{1}{a}\int t^n dt = \frac{t^{n+1}}{a(n+1)} + C$

$$= \frac{[f(ax+b)]^{n+1}}{a(n+1)} + C.$$

where C is any arbitrary constant.

18.

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

Ans - Given expression is, $\dfrac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

$\therefore \dfrac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \dfrac{1}{\sqrt{\sin^3 x \sin x \cos\alpha + \cos\alpha \sin x}}$

$= \dfrac{1}{\sqrt{\sin^4 x \cos\alpha + \sin^3 x \sin\alpha \cos x}}$

$= \dfrac{1}{\sin^2 x \sqrt{\cos\alpha + \sin\alpha \cot x}} = \dfrac{\csc^2 x}{\sqrt{\cos\alpha + \sin\alpha \cot x}}$

Now, substitute $\cos\alpha + \sin\alpha \cot x = t$

$\therefore -\csc^2 x \sin\alpha \, dx = dt$

$\therefore \int \dfrac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx = \int \dfrac{\csc^2 x}{\sqrt{\cos\alpha + \sin\alpha \cot x}} = \dfrac{-1}{\sin\alpha} \int \dfrac{dt}{\sqrt{t}}$

$= \dfrac{-2\sqrt{t}}{\sin\alpha} + C$

$\Rightarrow \int \dfrac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx = \dfrac{-2\sqrt{\cos\alpha + \sin\alpha \cot x}}{\sin\alpha} + C$

$= \dfrac{-2}{\sin\alpha} \sqrt{\cos\alpha + \dfrac{\sin\alpha \cos x}{\sin x}} + C$

where C is an arbitrary constant.

19.

$\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$

Ans - Given expression is $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Let $I = \int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

Now, substitute $x = \cos^2\theta$

$\therefore dx = -2\sin\theta\cos\theta\,dt$

$\therefore I = \int \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}\,(-2\sin\theta\cos\theta)d\theta$

$$= -\int \sqrt{\dfrac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\,\sin 2\theta\, d\theta$$

$$= -\int \tan\frac{\theta}{2}\,2\sin\theta\cos\theta\, d\theta$$

$\Rightarrow I = -2\int \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\theta\, d\theta$

$$= -4\int \sin^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right) d\theta$$

$\Rightarrow I = -8\int \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\,d\theta + 4\int \sin^2\frac{\theta}{2}\,d\theta$

$$= 2\int \sin^2\frac{\theta}{2}\,d\theta + 4\int \sin^2\frac{\theta}{2}\,d\theta$$

$\Rightarrow I = -2\int \left(\dfrac{1-\cos 2\theta}{2}\right)d\theta + 4\int \left(\dfrac{1-\cos\theta}{2}\right)d\theta$

$$= -2\left[\dfrac{\theta}{2}-\dfrac{\sin 2\theta}{2}\right] + 4\left[\dfrac{\theta}{2}-\dfrac{\sin\theta}{2}\right] + C$$

$\Rightarrow I = -\theta + \dfrac{\sin 2\theta}{2} + 2\sin\theta + C$

$$= \theta + \sqrt{1-\cos^2\theta}\cos\theta - 2\sqrt{1-\cos^2\theta} + C$$

$\Rightarrow I = \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} - 2\sqrt{1-x} + C$

$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$

where C is any arbitrary constant.

20.

$$\frac{2 + \sin 2x}{1 + \cos 2x} e^x$$

Ans - Given expression is, $\frac{2+\sin 2x}{1+\cos 2x} e^x$.

Let $I = \int \dfrac{2 + \sin 2x}{1 + \cos 2x} e^x dx$

$\Rightarrow I = \int \dfrac{2 + 2\sin x \cos x}{2\cos^2 x} e^x dx = \int \left(\dfrac{1 + \sin x \cos x}{\cos^2 x}\right) e^x dx$

$\qquad\qquad = \int (\sec^2 x + \tan x) e^x dx$

Now, consider $f(x) = \tan x$

$\therefore f'(x) = \sec^2 x\, dx$

$\therefore I = \int \dfrac{2 + \sin 2x}{1 + \cos 2x} e^x dx = \int \big(f(x) + f'(x)\big) e^x dx = e^x f(x) + C$

$\qquad\qquad = e^x \tan x + C$

where C is any arbitrary constant.

21.

$$\frac{x^2 + x + 1}{(x + 1)^2 (x + 2)}$$

Ans - Given expression is, $\dfrac{x^2 + x + 1}{(x + 1)^2(x + 2)}$

$\therefore \dfrac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \dfrac{A}{(x + 1)} + \dfrac{B}{(x + 1)} + \dfrac{C}{(x + 2)}$ ... (1)

$\Rightarrow x^2 + x + 1 = A(x + 1)(x + 2) + B(x + 2) + C(x + 1)^2$

$\Rightarrow x^2 + x + 1 = A(x^2 + 3x + 2) + B(x + 2)$
$\qquad\qquad\qquad + C(x^2 + 2x + 1)$

On equating coefficients of x^2, x and constant term, we get,

$A + C = 1$ ... (2)

$3A + B + 2C = 1$ ... (3)

$2A + 2B + C = 1$ ... (4)

On solving these equations, values of A, B , C will be

$A = -2, \quad B = 1, \quad C = 3$

Now, from equation (1) it can be clearly obtained that

$\int \dfrac{x^2 + x + 1}{(x + 1)^2(x + 2)} dx = \int \left\{ \dfrac{-2}{(x + 1)} + \dfrac{1}{(x + 1)} + \dfrac{3}{(x + 2)} \right\} dx$

$= -2\int \dfrac{1}{x + 1} dx + \int \dfrac{1}{(x + 1)^2} dx + 3\int \dfrac{1}{x + 2} dx$

$= -2\log|x + 1| + 3\log|x + 2| - \dfrac{1}{(x + 1)} + C$

Where C is any arbitrary constant.

22.

$\tan^{-1} \sqrt{\dfrac{1 - x}{1 + x}}$

Ans - Given expression is, $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$

Let $I = \int \tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\,dx$

Now, consider $x = \cos\theta$

$\therefore dx = -\sin\theta\,d\theta$

$\therefore I = \int \tan^{-1}\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}(-\sin\theta)d\theta$

$$= -\int \tan^{-1}\sqrt{\dfrac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\,\sin\theta\,d\theta$$

$$= -\int \tan^{-1}\tan\frac{\theta}{2}\,\sin\theta\,d\theta$$

$\Rightarrow I = -\dfrac{1}{2}\int \theta\sin\theta\,d\theta = -\dfrac{1}{2}\left[\theta(-\cos\theta) - \int 1.(-\cos\theta)d\theta\right]$

$$= -\dfrac{1}{2}\left[\theta(\cos\theta) + \sin\theta\right]$$

$\Rightarrow I = \dfrac{x}{2}\cos^{-1}x - \dfrac{1}{2}\sqrt{1-x^2} + C$

$$= \dfrac{1}{2}\left(x\cos^{-1}x - \sqrt{1-x^2}\right) + C$$

where C is any arbitrary constant.

23.

$$\dfrac{\sqrt{x^2+1}[\log(x^2+1) - 2\log x]}{x^4}$$

Ans - Given expression is, $\dfrac{\sqrt{x^2+1}\,[\log(x^2+1)-2\log x]}{x^4}$

Let $I = \displaystyle\int \dfrac{\sqrt{x^2+1}\,[\log(x^2+1)-2\log x]}{x^4}\,dx$

$\Rightarrow I = \dfrac{\sqrt{x^2+1}}{x^4}\,[\log(x^2+1)-\log x^2]$

$\qquad\qquad = \dfrac{\sqrt{x^2+1}}{x^4}\left[\log\left(\dfrac{x^2+1}{x^2}\right)\right]$

Consider $1+\dfrac{1}{x^2} = t$

$= \dfrac{1}{x^3}\sqrt{\dfrac{x^2+1}{x^2}}\left[\log\left(1+\dfrac{1}{x^2}\right)\right]$

$\therefore \dfrac{-2}{x^3}\,dx = dt$

Integrate given expression as shown below

$\therefore I = \displaystyle\int \dfrac{\sqrt{x^2+1}\,[\log(x^2+1)-2\log x]}{x^4}\,dx$

$\qquad\qquad = \displaystyle\int \dfrac{1}{x^3}\sqrt{\dfrac{x^2+1}{x^2}}\left[\log\left(1+\dfrac{1}{x^2}\right)\right]dx$

$\qquad\qquad = \dfrac{-1}{2}\displaystyle\int t^{\frac{1}{2}}\log t\,dt + C$

Using integration by parts, it can be obtained that,

$$I = \frac{-1}{2}\left[\log t \cdot \int t^{\frac{1}{2}}dt - \left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$= \frac{-1}{2}\left[\log t \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$\Rightarrow I = \frac{-1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}}dt\right] = \frac{-1}{3}t^{\frac{3}{2}}\log t + \frac{2}{9}t^{\frac{3}{2}}$$

$$= \frac{-1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right]$$

$$\Rightarrow I = \frac{-1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t - \frac{2}{3}\int t^{\frac{1}{2}}dt\right] = \frac{-1}{3}t^{\frac{3}{2}}\log t + \frac{2}{9}t^{\frac{3}{2}}$$

$$= \frac{-1}{3}t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right]$$

$$\Rightarrow I = \frac{-1}{2}\left[1+\frac{1}{x^2}\right]\left(\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right) + C$$

where C is any arbitrary constant.

Evaluate the definite integrals in Exercises 24 to 31

24.

$$\int_{\frac{\pi}{5}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Ans - Assume, $I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$

$\Rightarrow I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - 2\sin \frac{x}{2} \cos \frac{x}{2}}{+2\sin^2 \frac{x}{2}} \right) dx$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \left(\frac{\csc 2\frac{x}{2}}{2} - \cot \frac{x}{2} \right) dx$

Now, substitute $f(x) = -\cot \frac{x}{2}$

$\Rightarrow f'(x) = -\left(-\frac{1}{2} \csc^2 \frac{x}{2} \right) dx = \frac{1}{2} \csc^2 \frac{x}{2} \, dx$

$\Rightarrow I = \int_{\frac{\pi}{2}}^{\pi} e^x (f(x) + f'(x)) dx = [e^x f(x)]_{\frac{\pi}{2}}^{\pi} = [e^x \cot \frac{x}{2}]_{\frac{\pi}{2}}^{\pi}$

$\Rightarrow I = [e^x \cot \frac{x}{2} - e^{\frac{\pi}{2}} \cot \frac{\pi}{4}] = [0 - e^{\frac{\pi}{2}}] = -e^{\frac{\pi}{2}}$

25.

$\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$

Ans - Given expression is, $\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$.

Assume, $I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{\sin x \cos x}{\cos^4 x}}{\frac{\sin^4 x + \cos^4 x}{\cos^4 x}} dx = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$

Substitute $\tan^2 x = t$

Differentiating on both sides,

$\therefore 2\tan x \sec^2 x \, dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{4}, t = 1$.

$\therefore I = \frac{1}{2} \int_0^1 \frac{dt}{1 + t^2} = \frac{1}{2} [\tan^{-1} t]_0^1$

$= \frac{1}{2} [\tan^{-1}(1) - \tan^{-1}(0)] = \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8}$

26.

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$$

Ans - Given expression is, $\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$.

Assume, $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4\sin^2 x}\, dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1 - \cos^2 x)}\, dx$$

$$= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 4 - 3\cos^2 x}{-3\cos^2 x + 4}\, dx$$

$$\Rightarrow I = -\frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3\cos^2 x}{4 - 3\cos^2 x}\, dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4}{4 - 3\cos^2 x}\, dx$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4\sec^2 x - 3}\, dx$$

$$\Rightarrow I = \frac{-1}{3} [x]_0^{\frac{\pi}{2}} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1 + \tan^2 x) - 3}\, dx$$

$$= \frac{-\pi}{6} + \frac{2}{3} \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{(1 + 4\tan^2 x)}\, dx \ ... \ (1)$$

Observe, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{(1 + 4\tan^2 x)}\, dx$

Now, substitute $2\tan x = t$

$\therefore 2\sec^2 x\, dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \infty$.

$$\therefore \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{(1 + 4\tan^2 x)}\, dx = \int_0^{\infty} \frac{dt}{(1 + t^2)}\, dx = [\tan^{-1}(t)]_0^{\infty}$$

$$= [\tan^{-1}(\infty) - \tan^{-1}(0)] = \frac{\pi}{2}$$

Henceforth from equation (1), we get,

$$I = -\frac{\pi}{6} + \frac{2}{3}\left(\frac{\pi}{2}\right) = -\frac{\pi}{6} + \frac{2\pi}{6} = \frac{\pi}{6}$$

27.

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Ans - Given expression is, $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$.

Assume, $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x+\cos x}{\sqrt{\sin 2x}} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin x \cos x)}} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{1-(\sin^2 x + \cos^2 x - 2\sin x \cos x)}} dx$

$\Rightarrow I = \int_0^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}} dx$

Now, substitute $(\sin x - \cos x) = t$

$\therefore (\sin x + \cos x)dx = dt$

When $x = \frac{\pi}{6}, t = \left(\frac{1-\sqrt{3}}{2}\right)$ and when $x = \frac{\pi}{3}, t = \left(\frac{\sqrt{3}-1}{2}\right)$

$\therefore I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} = \int_{-\left(\frac{-1+\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$

As $\frac{1}{\sqrt{1-(-t)^2}} = \frac{1}{\sqrt{1-t^2}}$, it can be thus obtained that $\frac{1}{\sqrt{1-t^2}}$ is an even function,

$\therefore \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

$\therefore I = 2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} = [2\sin^{-1} t]_0^{\frac{\sqrt{3}-1}{2}} = 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$

28.

$$\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$$

Ans - Given expression is, $\int_0^1 \dfrac{dx}{\sqrt{1+x}-\sqrt{x}}$.

Assume, $I = \int_0^1 \dfrac{dx}{\sqrt{1+x}-\sqrt{x}}$

$\Rightarrow I = \int_0^1 \dfrac{1}{\sqrt{1+x}-\sqrt{x}} \times \dfrac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} dx$

$\qquad = \int_0^1 \dfrac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$

$\Rightarrow I = \int_0^1 \sqrt{1+x}\,dx + \int_0^1 \sqrt{x}\,dx = \dfrac{2}{3}\left[(1+x)^{\frac{2}{3}}\right]_0^1 + \dfrac{2}{3}\left[(x)^{\frac{3}{2}}\right]_0^1$

$\qquad = \dfrac{2}{3}\left[(2)^{\frac{2}{3}} - 1\right] + \dfrac{2}{3} = \dfrac{4\sqrt{2}}{3}$

29.

$\int_0^{\frac{\pi}{4}} \dfrac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$

Ans - Given expression is, $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$.

Assume, $I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

Now, substitute $\sin x - \cos x = t$

$\therefore (\cos x + \sin x)dx = dt$

When $x = 0, t = -1$ and when $x = \frac{\pi}{4}, t = 0$

$\therefore (\sin x - \cos x)^2 = t^2$

$\Rightarrow 1 - 2\sin x\cos x = t^2$

$\Rightarrow 1 - \sin 2x = t^2$

$\Rightarrow \sin 2x = 1 - t^2$

$\therefore I = \int_{-1}^0 \frac{dt}{9 + 16(1 - t^2)} = \int_{-1}^0 \frac{dt}{25 - 16t^2}$

$$= \int_{-1}^0 \frac{dt}{(5)^2 - (4t)^2}$$

$\Rightarrow I = \frac{1}{4}\left[\frac{1}{2(5)} \log \left|\frac{5 + 4t}{5 - 4t}\right|\right]_{-1}^0 = \frac{1}{40}\left[\log |1| - \log \left|\frac{1}{9}\right|\right]$

$$= \frac{1}{40} \log |9|$$

30.

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Ans – Given expression is, $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$

Assume, $I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$

$\qquad = \int_0^{\frac{\pi}{2}} \sin x \cos x \tan^{-1}(\sin x) dx$

Substitute $\sin x = t \Rightarrow \cos x\, dx = dt$

When $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = 1$

$\Rightarrow I = 2 \int_0^1 t \tan^{-1}(t) dt$

On equating coefficients of x^2, x and constant term, we get,

$A + C = 0 \,...........(2)$

$A + B = 0 \,...........(3)$

$B = 1 \,...(4)$

On solving these equations, values of A, B, C will be,

$A = -1, B = 1, C = 1$

From equation (1) we get,

$I = \int_1^3 \frac{dx}{x^2(x+1)} = \int_1^3 \left\{ \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$

$\Rightarrow I = \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3 = \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3$

$\qquad = \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log(2) + 1$

$\Rightarrow I = \log 4 - \log 3 - \log 2 + \frac{2}{3} = \log 2 - \log 3 + \frac{2}{3}$

$\qquad = \log\left(\frac{2}{3}\right) + \frac{2}{3}$

Hence proved.

31.

$$\int_1^4 [|x-1| + |x-2| + |x-3|]\, dx$$

Ans - Given expression is $\int_1^4 [|x-1| + |x-2| + |x-3|]\, dx$,

Assume, $\int_1^4 [|x-1| + |x-2| + |x-3|]\, dx$

$$\Rightarrow I = \int_1^4 |x-1|\, dx + \int_1^4 |x-2|\, dx + \int_1^4 |x-3|\, dx$$

$\therefore I = I_1 + I_2 + I_3 \ldots (1)$

$$I_1 = \int_1^4 |x-1|\, dx, I_2 = \int_1^4 |x-2|\, dx, I_3 = +\int_1^4 |x-3|\, dx$$

Let $I_1 = \int_1^4 |x-1|\, dx$, where $(x-1) \geq 0 \vee 1 \leq x \leq 4$

$$\therefore I_1 = \int_1^4 (x-1)\, dx = \left[\frac{x^2}{2} - x\right]_1^4 = \left[8 - 4 - \frac{1}{2} + 1\right]$$

$$= \frac{9}{2} \ldots (2)$$

Again, $I_2 = \int_1^4 |x-2|\, dx$, where $(x-2) \geq 0 \vee 2 \leq x \leq 4$ and $(x-2) \leq 0 \vee 1 \leq x \leq 2$.

$$\therefore I_2 = \int_1^2 (2-x)\, dx + \int_2^4 (x-2)\, dx$$

$$= \left[2x - \frac{x^2}{2}\right]_1^4 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$\Rightarrow I_2 = \left[4 - 2 - 2 + \frac{1}{2}\right] + [8 - 8 - 2 + 4] = \frac{1}{2} + 2 = \frac{5}{2} \#(3)$$

Also, $I_3 = \int_1^4 |x-3|\, dx$, where $(x-3) \geq 0 \vee 3 \leq x \leq 4$ and $(x-3) \leq 0 \vee 1 \leq x \leq 3$.

$$\therefore I_3 = \int_1^3 (3-x)\, dx + \int_3^4 (x-3)\, dx$$

$$= \left[3 - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right] = 2 + \frac{1}{2}$$

$$= \frac{5}{2} \dots (4)$$

Now, from equations (1), (2), (3) and (4) it can be obtained that,

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

$$\Rightarrow 2I = \pi \int_0^\pi \frac{\sin x + 1 - 1}{1 + \sin x} dx$$

$$= \pi \int_0^\pi dx - \pi \int_0^\pi \frac{1}{1 + \sin x} dx$$

$$= \pi [x]_0^\pi - \pi \int_0^\pi \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi [x]_0^\pi - \pi \int_0^\pi (\sec^2 x - \tan x \sec x) dx$$

$$= \pi^2 - \pi [\tan x - \sec x]_0^\pi$$

$$\Rightarrow 2I = \pi^2 - \pi [0 - (-1) - 0 + 1] = \pi^2 - 2\pi$$

$$\Rightarrow I = \frac{\pi(\pi - 2)}{2}$$

Prove the following (Exercises 32 to 37)

32.

$$\int_1^3 \frac{dx}{x^2(x + 1)} = \frac{2}{3} + \log \frac{2}{3}$$

Ans - Given equation is, $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log \frac{2}{3}$.

Assume, $\int_1^3 \frac{dx}{x^2(x+1)}$

Now consider it as shown below,

$$\therefore \frac{1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x + 1)} \dots \dots (1)$$

$$\Rightarrow 1 = Ax(x + 1) + B(x + 1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

On equating coefficients of x^2, x and constant term, we get,

$A + C = 0$ ... ... (2)

$A + B = 0$ ... ... (3)

$B = 1$ ... (4)

On solving these equations, values of A, B, C will be,

$A = -1, B = 1, C = 1$

From equation (1) we get,

$$I = \int_1^3 \frac{dx}{x^2(x+1)} = \int_1^3 \left\{ \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$\Rightarrow I = \left[-\log x - \frac{1}{x} + \log (x+1) \right]_1^3 = \left[\log \left(\frac{x+1}{x} \right) - \frac{1}{x} \right]_1^3$$

$$= \log \left(\frac{4}{3} \right) - \frac{1}{3} - \log (2) + 1$$

$$\Rightarrow I = \log 4 - \log 3 - \log 2 + \frac{2}{3} = \log 2 - \log 3 + \frac{2}{3}$$

$$= \log \left(\frac{2}{3} \right) + \frac{2}{3}$$

Hence proved.

33.

$\int_0^4 xe^x dx = 1$

Ans - Given equation is, $\int_0^4 xe^x dx = 1$.

Assume, $I = \int_0^4 xe^x dx$

Using integration by parts, it can be obtained that,

$$I = x \int_0^4 e^x dx - \int_0^4 \left\{ \left(\frac{d(x)}{dx} \right) \int e^x \right\} = [xe^x]_0^1 - [e^x]_0^1$$

$$= e - e + 1 = 1$$

Hence proved.

34.

$$\int_1^{-1} x^{17}\cos^4 x\, dx = 0$$

Ans - Given equation is, $\int_1^{-1} x^{17}\cos^4 x\, dx = 0$.

Assume, $I = \int_1^{-1} x^{17}\cos^4 x\, dx$

Now, consider $f(x) = x^{17}\cos^4 x$

$\therefore f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

$\Rightarrow f(x)$ is an odd function

$\Rightarrow$ It is known to us that when $f(x)$ is an odd function, then

$\int_{-a}^a f(x)dx = 0$

$\therefore I = \int_1^{-1} x^{17}\cos^4 x\, dx = 0$

Hence proved.

35.

$$\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}$$

Ans - Given equation is, $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \frac{2}{3}$.

Assume, $I = \int_0^{\frac{\pi}{2}} \sin^3 x\, dx$

$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x \sin x\, dx = \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\sin x\, dx$

$$= \int_0^{\frac{\pi}{2}} \sin x\, dx - \int_0^{\frac{\pi}{2}} \cos^2 x \sin x\, dx$$

$\Rightarrow I = [-\cos x]_0^{\frac{\pi}{2}} + \left[\dfrac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}} = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

Hence proved.

36.

$$\int_0^{\frac{\pi}{4}} 2\tan^3 x\, dx = 1 - \log 2$$

Ans - Given equation is, $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$.

Assume, $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx$

$\Rightarrow I = \int_0^{\frac{\pi}{4}} 2\tan^2 x \tan x dx = \int_0^{\frac{\pi}{4}} (1 - \sec^2 x)\tan x dx$

$$= \int_0^{\frac{\pi}{4}} \tan x dx - \int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx$$

$\Rightarrow I = 2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}} + 2[\log \cos x]_0^{\frac{\pi}{4}}$

$$= 1 + 2\left[\log \cos \frac{\pi}{4} - \log \cos 0\right]_0^{\frac{\pi}{2}}$$

$$= 1 - \log 2 - \log 1$$

$\Rightarrow I = 1 - \log 2$

Hence proved.

37.

$$\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$$

Ans - Given equation is, $\int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - 1$.

Assume, $I = \int_0^1 \sin^{-1} x \, dx$

$\Rightarrow I = \int_0^1 \sin^{-1} x \cdot 1 \, dx$

Using integration by parts, we get

$I = [\sin^{-1} x \cdot x]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} x \, dx$

$= [x \sin^{-1} x]_0^1 + \frac{1}{2} \int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} \, dx$

Substitute $1 - x^2 = t$

$\therefore (-2x) dx = dt$

When $x = 0, t = 1$ and when $x = 1, t = 0 - 1$

$\therefore I = [x \sin^{-1} x]_0^1 + \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{t}} = [x \sin^{-1} x]_0^1 + \frac{1}{2} [2\sqrt{t}]_1^0$

$= \sin^{-1} 1 - \sqrt{1} = \frac{\pi}{2} - 1$

Hence proved.