NCERT Solutions for Class 12 Maths Chapter 6 – Application of Derivatives

Exercise 6.1

1.

Find the rate of change of the area of a circle with respect to its radius r when
(a) r = 3 cm **(b) r = 4 cm**

Ans - Area of a circle(A) with radius (r) is $A = \pi r^2$

For a given radius, change in area of circle is given by,

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

(a) When r = 3cm

$$\frac{dA}{dr} = 2\pi(3) = 6\pi$$

As a result, when radius of circle is 3cm, the area of the circle changes at a rate of 6π cm

(b) When r = 4cm

$$\frac{dA}{dr} = 2\pi(4) = 8\pi$$

As a result, when radius of circle is 4cm, the area of the circle changes at a rate of 8π cm

2.

The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?

Ans - Let x be the length of a side, v be the volume, and s be the surface area of the cube.

When x is a function of time t, we have

$V = x^3$ and $S = 6x^2$

Given that $\frac{dV}{dt} = 8cm^3/s$

By chain rule we get,

$$\frac{d}{dt}(x^3) = 8$$

$$\Rightarrow 3x^2.\frac{dx}{dt} = 8$$

$$\Rightarrow \frac{dx}{dt} = \frac{8}{3x^2} \qquad \ldots.(1)$$

Rate of change in surface are of cube $= \frac{dy}{dt} = \frac{d}{dt}(6x^2)$

$$\Rightarrow \frac{dy}{dt} = 6\frac{d}{dt}(x^2) = 6\left(2x\frac{dx}{dt}\right) = 12x\left(\frac{dx}{dt}\right)$$

Substitute value of equation (1)

$$\Rightarrow \frac{dy}{dt} = 12x\left(\frac{8}{3x^2}\right) = 4\left(\frac{8}{x}\right) = \frac{32}{x}$$

Thus, when x = 12cm,

$$\frac{dS}{dt} = \frac{32}{12}cm^2/s$$

$$= \frac{8}{3}\ cm^2/s$$

As a result, if the cube's edge length is 12cm, the surface area is rising at a rate of $= \frac{8}{3}\ cm^2/s$.

3.

The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

Ans - For radius (r), the area of a circle (A) is $A = \pi r^2$.

For time (t), the rate of change of area (A) is given by,

$$\frac{dA}{dt} = \frac{d}{d}(\pi r^2)$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad \text{(By chain rule)}$$

Given that the increase in radius of circle is,

$$\frac{dr}{dt} = 3\text{cm/s}$$

$$\frac{dA}{dt} = 2\pi r(3) = 6\pi r$$

So, when $r = 10\text{cm}$

$$\frac{dA}{dt} = 6\pi(10) = 60\pi\ \text{cm}^2/\text{s}$$

As a result, when the radius of the circle is **10cm**, the rate at which the area of the circle increase is $\mathbf{60\pi\ cm^2/s}$

4.

An edge of a variable cube is increasing at the rate of 3cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

Ans - Let x be length of a side and V be volume of the cube.
Then, $V = x^3$

$$\therefore \frac{dV}{dt} = 3x^2.\frac{dx}{dt} \qquad \text{(By chain rule)}$$

Given that,

$$\frac{dx}{dt} = 3\text{cm/s}$$

$$\therefore \frac{dV}{dt} = 3x^2(3) = 9x^2$$

So, when $x = 10\text{cm}$,

$$\frac{dV}{dt} = 9(10)^2 = 900\ \text{cm}^3/\text{s}$$

As a result, when the edge is **10cm** long, the volume of the cube increases at the rate of $\mathbf{900\ cm^3/s}$.

5.

A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

Ans - Area of a circle (A) with radius (r) is given by, $A = \pi r^2$

For time (t) the rate of change of area (A) is given by,

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2)$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{(By chain rule)}$$

Given that, $\frac{dr}{dt} = 5cm/s$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \times 5 = 10\pi r$$

So, when $r = 8cm$

$$\Rightarrow \frac{dA}{dt} = 10\pi(8) = 80\pi$$

As a result, when the circle wave's radius is **8cm**, the enclosed area grows at a pace of $\mathbf{80cm^3/s}$.

6.

The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

Ans - Circumference of a circle (C) with radius (r) is given by $C = 2\pi r$

Rate of change of circumference (C) for time (**t**) is given by,

$$\frac{dC}{dt} = \frac{d}{dr}(2\pi r) = 2\pi \frac{dr}{dt} \qquad \text{(By chain rule)}$$

Given that, $\frac{dr}{dt} = 0.7cm/s$

$$\Rightarrow \frac{dC}{dt} = 2\pi(0.7) = 1.4\pi \ cm/s$$

Hence, rate of increase of the circumference is $\mathbf{1.4\pi cm/s}$

7.

The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8cm$ and $y = 6cm$, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

Ans - Given that length x is decreasing at the rate of 5cm/min and width y is increasing at the rate of 4cm/min

$$\frac{dx}{dt} = -5cm/min,$$

$$\frac{dy}{dt} = 4cm/min$$

(a) Perimeter (P) of a rectangle is given by

$$P = 2(x + y)$$

$$\therefore \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$= -2cm/min$$

Hence, the rate of decrease in the perimeter of rectangle is **2cm/min**.

(b) Area(A) of a rectangle is given by,

$$A = x \times y$$

$$\frac{dA}{dt} = \frac{dx}{dt}.y + x.\frac{dy}{dt}$$

$$= -5y + 4x$$

When $x = 8cm$ and $y = 6cm$,

$$\frac{dA}{dt} = (-5 \text{ x } 6 + 4 \text{ x } 8)cm^2/min$$

$$= 2cm^2/min$$

Hence, the rate of increase in the area of the rectangle is **$2cm^2/min$**

8.

A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

Ans - Volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

For time (t), the rate of change of volume (V) is given by,

$$\frac{dV}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dt} = 4\pi r^2 . \frac{dr}{dt}$$

Given that, $\frac{dV}{dt} = 900 cm^3/s$

$$\therefore 900 = 4\pi r^2 . \frac{dr}{dt}$$

$$\Longrightarrow \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

So, when the radius $r = 15cm$,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi}$$

Hence, the rate at which the balloon's radius increase when the radius is **15cm** is $\frac{1}{\pi}$ **cm/sec**.

9.

A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

Ans - Volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

Rate of change of volume (V) for time (**t**) is given by,

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi \times 3r^2$$

$$\Rightarrow \frac{dV}{dr} = 4\,\pi r^2$$

So, when radius $r = 10\text{cm}$,

$$\frac{dV}{dr} = 4\,\pi(10)^2$$

$$= 400\pi$$

Hence, the rate of increase in the volume of the balloon is $\mathbf{400\pi\ cm^3/sec}$

10.

A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

Ans - Let height of the wall be y m and foot of the ladder be x m away from the wall.

By Pythagoras Theorem, we have:

$x^2 + y^2 = 25$ (Length of the ladder is $\mathbf{5m}$)

$\Rightarrow y = \sqrt{25 - x^2}$

Then, the rate of change of height $\mathbf{y}$ for time $\mathbf{t}$ is given by,

$$\frac{dy}{dt} = \frac{-x}{\sqrt{25 - x^2}} \cdot \frac{dx}{dt}$$

Given that, $\frac{dx}{dt} = 2\text{cm/s}$

$$\therefore \frac{dy}{dt} = \frac{-2x}{\sqrt{25 - x^2}}$$

So, when $x = 4$ m we have:

$$\frac{dy}{dt} = \frac{-2(4)}{\sqrt{25 - 4^2}} = \frac{8}{3}$$

Hence, the rate of decrease in the height of the ladder on the wall is $\frac{8}{3}\,\mathbf{cm/sec}$.

11.

A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate

Ans - Given the equation of the curve as $6y = x^3 + 2$

Rate of change of position of particle for time (t) is given by,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt} + 0$$

$$\Rightarrow 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

When the particle's y-coordinate changes 8 times as fast as its x-coordinate i.e.,$\left(\frac{\mathbf{dy}}{\mathbf{dt}} = \frac{\mathbf{8dx}}{\mathbf{dt}}\right)$, we have:

$$2\left(8\frac{dx}{dt}\right) = x^2\frac{dx}{dt}$$

$$\Rightarrow 16\frac{dx}{dt} = x^2\frac{dx}{dt}$$

$$\Rightarrow (x^2 - 16)\frac{dx}{dt} = 0$$

$$\Rightarrow x^2 = 16$$

$$\Rightarrow x = \pm 4$$

Where $x = -4$,

$$y = \frac{-13}{3}$$

Where $x = 4$,

$$y = 11$$

Hence, the points required on the curve are $\mathbf{(4,11)}$ and $\left(\mathbf{-4, \frac{-13}{3}}\right)$

12.

The radius of an air bubble is increasing at the rate of $\frac{1}{2}$cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

Ans - Air bubble is in the shape of a sphere. The volume of an air bubble (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^3$$

Rate of change of volume (V) for time (**t**) is given by,

$$\frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt} \quad \text{(By chain rule)}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right).\frac{dr}{dt}$$

$$= 4\pi r^2.\frac{dr}{dt}$$

Given that, $\frac{dr}{dt} = \frac{1}{2}\text{cm/s}$

So, when $\mathbf{r = 1cm,}$

$$\frac{dV}{dt} = 4\,\pi(1)^2\left(\frac{1}{2}\right) = 2\pi cm^3/s$$

Hence, the volume of the bubble increases at the rate of $\mathbf{2\pi cm^3}/s$

13.

A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

Ans - Volume of a sphere (V) with radius (r) is given by,

$$V = \frac{4}{3}\pi r^2$$

Given diameter is $\frac{3}{2}(2x + 1)$

$$\Rightarrow r = \frac{3}{4}(2x + 1)$$

$$V = \frac{4}{3}\pi r^3 = \frac{9}{16}\pi(2x + 1)^3$$

Hence, rate of change of volume with respect to x will be

$$\frac{dV}{dx} = \frac{9}{16}\pi\frac{d}{dt}(2x + 1)^3$$

$$\Rightarrow \frac{dV}{dx} = \frac{9}{16}\pi.3(2x + 1)^2\frac{d}{dt}(2x + 1)$$

$$\Rightarrow \frac{dV}{dx} = \frac{27}{16}\pi.(2x + 1)^2.2$$

$$\Rightarrow \frac{dV}{dx} = \frac{27}{8}\pi(2x + 1)^2$$

14.

Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Ans - Volume of a sphere (V) with radius(r) and height(h) is given by,

$$V = \frac{1}{3}\pi r^2 h$$

Given that, $h = \frac{1}{6}r$

$$\Rightarrow r = 6h$$

$$\therefore V = \frac{1}{3}\pi(6h)^2 h = 12\ \pi h^3$$

Rate of change of volume for time (t) is given by

$$\frac{dV}{dt} = 12\pi\frac{d}{dh}(h^3).\frac{dh}{dt} \qquad \text{(By chain rule)}$$

$$\Rightarrow \frac{dV}{dt} = 12\pi(3h^2)\frac{dh}{dt} = 36\pi h^2\frac{dh}{dt}$$

Given that, $\frac{dV}{dt} = 12\ cm^2/s$

So, when $h = 4\ cm/s$ we have:

$$12 = 36\pi(4)^2\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{12}{36\pi(16)}$$

$$\frac{dh}{dt} = \frac{1}{48\pi}$$

Hence, when the height of the sand cone is **4cm**, its height is increasing at the rate of $\mathbf{\frac{1}{48}}$ **cm/sec.**

15.

The total cost C(x) in Rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$.

Ans - Rate of change in total cost for output is known as marginal cost.

$$MC = \frac{dC}{dx} = 0.007(3x^2) - 0.003(2x) + 15$$

$$= 0.021x^2 - 0.006x + 15$$

When $x = 17$,

$$MC = 0.021(17)^2 - 0.006(17) + 15$$

$$= 0.021(289) - 0.006(17) + 15$$

$$= 6.069 - 0.102 + 15$$

$$= 20.967$$

Hence, the marginal cost when 17 units ae produced is **Rs.20.967**

16.

The total revenue in Rupees received from the sale of x units of a product is given by
$R(x) = 13x^2 + 26x + 15$
Find the marginal revenue when $x = 7$.

Ans - Rate of change in total cost for the number of units sold is known as marginal cost. Let the number of units sold be x.

$$MR = \frac{dR}{dx} = 13(2x) + 26$$

$$= 26x + 26$$

When $x = 7$,

$$MR = 26(7) + 26 = 208$$

Hence, the required marginal revenue is **Rs.208**

17.

The rate of change of the area of a circle with respect to its radius r at r = 6 cm is
(A) 10π (B) 12π (C) 8π (D) 11π

Ans - Area of a circle (A) with radius (r) is given by, $A = \pi r^2$

As a result, the area's rate of change in relation to its radius r is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$$

So, when $r = 6cm$,

$$\frac{dA}{dr} = 2\pi \text{ x } 6$$

$= 12\pi \text{ cm}^2/\text{s}$

∴ Required rate of change of the area of a circle is $12\pi \text{cm}^2/\text{s}$

Hence, **option B** is correct.

18.

The total revenue in Rupees received from the sale of x units of a product is given by
$R(x) = 3x^2 + 36x + 5$. The marginal revenue, when x = 15 is
(A) 116 (B) 96 (C) 90 (D) 126

Ans - Rate of change in total cost for the number of units sold is known as marginal cost.

$$\therefore MR = \frac{dR}{dx} = 6x + 36$$

So, when $x = 15$,

$MR = 6(15) + 36 = 90 + 36 = 126$

∴ Required marginal revenue is Rs.126

Hence, **option D** is correct.

Exercise 6.2

1.

Show that the function given by f (x) = 3x + 17 is increasing on R.

Ans - Let the two numbers in R be x_1 and x_2

$x_1 < x_2 \Rightarrow 3x_1 < 3x_2$

$\Rightarrow 3x_1 + 17 < 3x_2 + 17$

$= f(x_1) < f(x_2)$

Thus, the function is strictly increasing on **R**.

2.

Show that the function given by f (x) = e^{2x} is increasing on R.

Ans - Let the two numbers is R be x_1 and x_2

Then, we have:

$x_1 < x_2 \Rightarrow 2x_1 < 2x_2$

$\Rightarrow e^{2x_1} < e^{2x_2}$

$\Rightarrow f(x_1) < f(x_2)$

Hence, f is strictly increasing on **R**.

3.

Show that the function given by f (x) = sin x is

(a) increasing in $\left(0, \frac{\pi}{2}\right)$ **(b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$**

(c) neither increasing nor decreasing in (0, π)

Ans - Given function is $f(x) = \sin x$

$\therefore f'(x) = \cos x$

a) Since for each $x \in \left(0, \frac{\pi}{2}\right)$, $\cos x > 0$, we have $f'(x) > 0$.

Hence, f is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

b) Since for each $x \in \left(\frac{\pi}{2}, \pi\right)$, $\cos x < 0$, we have $f'(x) < 0$.

Hence, f is strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$

c) It is clear from the results obtained in (a) and (b) that f is neither increasing nor decreasing in $(0, n)$

4.

Find the intervals in which the function f given by
$f(x) = 2x^2 - 3x$ is
(a) increasing **(b) decreasing**

Ans – Given function is $f(x) = 2x^2 - 3x$

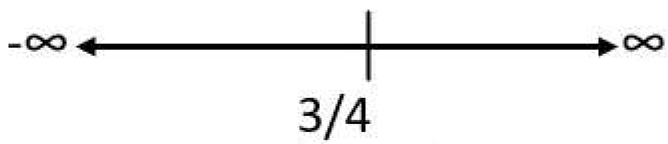
$\Rightarrow f'(x) = 4x - 3$

Now, $f'(x) = 0$

$$\Rightarrow x = \frac{3}{4}$$

Now, the point $\frac{3}{4}$ divides the real line into two disjoint intervals i.e.,

$\left(-\infty, \frac{3}{4}\right)$ and $\left(\frac{3}{4}, \infty\right)$



Number Line

In interval, $\left(-\infty, \frac{3}{4}\right)$, $f'(x) = 4x - 3 < 0$

Hence, the given function $(\mathbf{f})$ is strictly decreasing in interval $\left(-\infty, \frac{3}{4}\right)$

In interval, $\left(\frac{3}{4}, \infty\right)$, $f'(x) = 4x - 3 > 0$

Hence, the given function $(\mathbf{f})$ is strictly decreasing in interval $\left(\frac{3}{4}, \infty\right)$

5.

Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is
(a) increasing (b) decreasing

Ans - Given function is

$f(x) = 2x^3 - 3x^2 - 36x + 7$

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$

$= 6(x+2)(x-3)$

Now, $f'(x) = 0$

$\Rightarrow x = -2, 3$

The points $x = -2, 3$ divide the real line into three disjoints intervals

i.e., $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$

In interval, $(-\infty, -\mathbf{2})$, $(\mathbf{3}, \infty)$ $\mathbf{f}'(\mathbf{x})$ is positive while in the interval $(-\mathbf{2}, \mathbf{3})$ is negative.

Hence, the given function $(\mathbf{f})$ is strictly increasing in interval $(-\infty, -\mathbf{2}) \cup (\mathbf{3}, \infty)$ while the function $(\mathbf{f})$ is strictly decreasing in interval $(-\mathbf{2}, \mathbf{3})$.

6.

Find the intervals in which the following functions are strictly increasing or decreasing:

(a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$

(c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$

(e) $(x + 1)^3 (x - 3)^3$

Ans - a) Given that $f(x) = x^2 + 2x - 5$

$\therefore f'(x) = 2x + 2$

Now, $f'(x) = 0$

$\Rightarrow x = -1$

The real line is divided into two disjoint intervals $(-\infty, -1)$ and $(-1, \infty)$ by the point $x = -1$

In interval $(-\infty, -1)$,

$f'(x) = 2x + 2 < 0$

$\therefore$ (f) is strictly decreasing in the interval $(-\infty, -1)$

Thus (f) is strictly decreasing for $x < -1$

In interval $(-1, \infty)$,

$f'(x) = 2x + 2 > 0$

$\therefore$ (f) is strictly decreasing in the interval $(-1, \infty)$

Thus (f) is strictly increasing for $x > -1$

b) Given that, $f(x) = 10 - 6x - 2x^2$

$\therefore f'(x) = -6 - 4x$

Now, $f'(x) = 0$

$$\Rightarrow x = -\frac{3}{2}$$

Real line is divided into two disjoint intervals $\left(-\infty, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, \infty\right)$ by the point $x = -\frac{3}{2}$.

Interval $\left(-\infty, -\frac{3}{2}\right)$ i.e., when $x < -\frac{3}{2}$, $f'(x) = -6 - 4x < 0$.

$\therefore$ f is strictly increasing for $x < -\frac{3}{2}$

In interval $\left(-\frac{3}{2}, \infty\right)$ i.e., when $x > -\frac{3}{2}$, $f'(x) = -6 - 4x < 0$.

$\therefore$ f is strictly increasing for $x < -\frac{3}{2}$

c) Given that, $f(x) = -2x^3 - 9x^2 - 12x + 1$

$\therefore f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2)$

$= -6(x + 1)(x + 2)$

Now, $f'(x) = 0$

$\Rightarrow x = -1$ and $x = -2$

Real line is divided into three disjoint intervals $(-\infty, -2)$ and $(-2, -1)$ and $(-1, \infty)$ by the points

$x = -1, -2$ when $x < -2$ and $x > -1$

$\therefore f'(x) = -6(x + 1)(x + 2) < 0$

$\therefore$ f is strictly increasing for $x < -2 < x > -1$

Now, interval $(-2, -1)$ i.e., when $-2 < x < -1$

$f'(x) = -6(x + 1)(x + 2) > 0$

$\therefore$ f is strictly increasing for $-2 < x < -1$

d) Given that, $f(x) = 6 - 9x - x^2$

$\therefore f'(x) = -9 - 2x$

Now, $f'(x) = 0$

$\Rightarrow x = -\frac{9}{2}$

Real line is divided into two disjoint intervals $\left(-\infty, -\frac{9}{2}\right)$ and $\left(-\frac{9}{2}, \infty\right)$ by the point $x = -\frac{9}{2}$

In interval $\left(-\infty, -\frac{9}{2}\right)$ i.e., for $x < \frac{9}{2}$

$\therefore$ f is strictly increasing for $x < \frac{9}{2}$

In interval $\left(\frac{9}{2}, \infty\right)$ i.e., for $x < -\frac{9}{2}, f'(x) = -9 - 2x > 0$

$\therefore$ f is strictly increasing for $x < -\frac{9}{2}$

In interval $\left(-\frac{9}{2}, \infty\right)$ i.e., for $x > -\frac{9}{2}, f'(x) = -9 - 2x < 0$

$\therefore$ f is strictly increasing for $x > -\frac{9}{2}$

e) Given that, $f(x) = (x+1)^2 (x-3)^3$

$f'(x) = 3x(x+1)^2(x-3)^3 + 3(x-3)^2(x+1)^3$

$= 3(x+1)^2(x-3)^3[x-3+x+1]$

$= 6(x+1)^2(x-3)^3(x-1)$

Now, $f'(x) = 0$

$x = -1, 3, 1$

Points $x = -1, 1, 3$ divided the real line into four disjoint intervals. i.e. $(-\infty, -1)$, $(-1,1)$, $(1,3)$ and $(3, \infty)$

In interval $(-\infty, -1)$ and $(-1,1)$,

$f'(x) = 6(x+1)^2(x-3)^2(x-1) < 0$

$\therefore$ f is strictly decreasing in interval $(-\infty, -1)$ and $(-1,1)$

In interval $(1,3)$ and $(3, \infty)$,

$f'(x) = 6(x+1)^2(x-3)^2(x-1) > 0$

$\therefore$ f is strictly decreasing in the interval $(1,3)$ and $(3, \infty)$

7.

Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$, is an increasing function of x throughout its domain

Ans − Given that $y = \log(1 + x) - \dfrac{2x}{2 + x}$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = \frac{1}{1+\mathrm{x}} - \frac{(2+\mathrm{x})(2) - 2\mathrm{x}(1)}{(2+\mathrm{x})^2} = \frac{x^2}{(2+x)^2}$$

Now, $\dfrac{\mathrm{dy}}{\mathrm{dx}} = 0$

$$\Rightarrow \frac{x^2}{(2+x)^2} = 0$$

$$\Rightarrow x^2 = 0$$

$$\Rightarrow x = 0$$

Since $x > -1$ point $x = 0$ divides the domain $(-1, \infty)$ in two disjoint intervals i. e., $-1 < x < 0$ and $x > 0$

When $-1 < x < 0$ we have:

$x < 0 \Rightarrow x^2 > 0$

$x > -1 \Rightarrow (2 + x) > 0$

$\Rightarrow (2 + x)^2 > 0$

$$\therefore y = \frac{x^2}{(2+x)^2} > 0$$

Also, when $x > 0$

$x < 0 \Rightarrow x^2 > 0,$

$(2 + x)^2 > 0$

$$\therefore y = \frac{x^2}{(2+x)^2} > 0$$

Hence, the function f is increasing throughout this domain

8.

Find the values of x for which $\mathbf{y = [x(x-2)]^2}$ is an increasing function.

Ans - We have $y = [x(x-2)]^2 = [x^2 - 2x]^2$

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = 2(\mathrm{x}^2 - 2\mathrm{x})(2\mathrm{x} - 2) = 4x(x-2)(x-1)$$

Now, $\frac{dy}{dx} = 0$

$\Rightarrow 4x(x-2)(x-1) = 0$

$\Rightarrow x = 0,1,2$

Points $x = 0,1,2$ divide the real line into four disjoint intervals i.e.

$(-\infty, 0), (0,1), (1,2)$ and $(2, \infty)$.

In intervals $(-\infty, 0)$ and $(1, 2), \frac{dy}{dx} < 0$

$\therefore$ y is strictly decreasing in intervals $(-\infty, 0)\ and\ (1,2)$

However, in intervals $(0,1)$ and $(2, \infty), \frac{dy}{dx} > 0$

$\therefore$ y is strictly decreasing in intervals $(0,1)$ and $(2, \infty)$

$\therefore$ y is strictly decreasing in intervals $0 < x < 1$ and $x > 2$

9.

Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

Ans − We have, $y = \frac{4\sin\theta}{(2 + \cos\theta)} - \theta$

$$\therefore \frac{dy}{d\theta} = \frac{(2 + \cos\theta)(4\cos\theta) - 4\sin\theta(-\sin\theta)}{(2 + \cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2 + \cos\theta)^2} - 1 == \frac{8\cos\theta + 4}{(2 + \cos\theta)^2} - 1$$

Now. $\frac{dy}{d\theta} = 0$

$$\Rightarrow \frac{8\cos\theta + 4}{(2 + \cos\theta)^2} = 1$$

$$\Rightarrow 8\cos\theta + 4 = 4 + \cos^2\theta + 4\cos\theta$$

$$\Rightarrow \cos^2\theta - 4\cos\theta = 0$$

$$\Rightarrow \cos\theta = 0,4$$

Since $\cos\theta \neq 4$,

$\cos\theta = 0$

$$\Rightarrow \theta = \frac{\pi}{2}$$

Now, $\frac{dy}{d\theta} = \frac{\cos(4 - \cos\theta)}{(2 + \cos)^2}$

In the interval $\left[0, \frac{\pi}{2}\right]$,we have $\cos\theta > 0$

Also, $4 > \cos\theta \Rightarrow 4 - \cos\theta > 0$

$\therefore \cos\theta(4 - \cos 0) > 0$ and also $(2 + \cos\theta)^2 > 0$

$\Rightarrow \frac{\cos\theta(4 - \cos\theta)}{(2 + \cos)^2} > 0$

$\Rightarrow \frac{dy}{dx} > 0$

Therefore y is strictly increasing in the interval $\left(0, \frac{\pi}{2}\right)$.

Hence, y is increasing in the interval $\left(0, \frac{\pi}{2}\right)$

10.

Prove that the logarithmic function is increasing on (0, ∞).

Ans - Given function is f(x)logx

$f'(x) = \frac{1}{x}$

It is clear that for $x > 0, f'(x) = \frac{1}{x} > 0$

Hence, f(x)logx is strictly increasing in the interval $(0, \infty)$

11.

Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor decreasing on (– 1, 1)

Ans - Given function is $f(x) = x^2 - x + 1$

$\therefore f'(x) = 2x - 1$

Now, $f'(x) = 0$

$\Rightarrow x = \frac{1}{2}$

The point $\frac{1}{2}$ divides the interval (-1,1) into two disjoint intervals i.e.

$\left(-1, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$

Now, in the interval $\left(-1, \frac{1}{2}\right)$, $f'(x) = 2x - 1 < 0$

$\therefore$ f is strictly decreasing in the interval $\left(-1, \frac{1}{2}\right)$

However, in interval $\left(\frac{1}{2}, 1\right)$, $f'(x) = 2x - 1 > 0$

$\Rightarrow$ f is strictly decreasing in the interval $\left(\frac{1}{2}, 1\right)$

Hence, f is neither strictly increasing nor decreasing in interval $(-1, 1)$

12.

Which of the following function are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$

Ans - A) Let $f_1(x) = \cos x$

$\therefore f_1'(x) = -\sin x$

In interval, $\left(0, \frac{\pi}{2}\right)$, $f_1'(x) = -\sin x < 0$

$\therefore f_1(x) = \cos x$ is strictly decreasing in the interval $\left(0, \frac{\pi}{2}\right)$

B) Let $f_2(x) = \cos 2x$

$\therefore f_2'(x) = -2 \sin 2x$

Now,

$0 < x < \frac{\pi}{2} \Rightarrow 0 < 2x < \pi$

$\Rightarrow \sin 2x > 0 \Rightarrow -2 \sin 2x < 0$

$\therefore f_2'(x) = -2 \sin 2x < 0$ on $\left(0, \frac{\pi}{2}\right)$

$f_2(x) = \cos 2x$ is strictly decreasing in interval $\left(0, \frac{\pi}{2}\right)$

C) Let $f_3(x) = \cos 3x$

$\therefore f_3'(x) = -3 \sin 3x$

Now

$f_3{}'(x) = 0$

$\Rightarrow \sin 3x = 0$

$\Rightarrow 3x = \pi, x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow x = \frac{\pi}{3}$

The point $x = \frac{\pi}{3}$ divides the interval $\left(0, \frac{\pi}{2}\right)$ into two disjoint intervals i.e. $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

Now, in the interval $\left(0, \frac{\pi}{3}\right)$,

$f_3(x) = -3\sin 3x < 0 \left[0 < x < \frac{\pi}{3} \Rightarrow 0 < 3x < \pi\right]$

$\therefore f_3$ is strictly decreasing in interval $x\left(0, \frac{\pi}{3}\right)$

However, in the interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, $f_3(x) = -3\sin 3x > 0 \left[\frac{\pi}{3} < x < \frac{\pi}{2} \Rightarrow \pi < 3x < \frac{3\pi}{2}\right]$

$\therefore f_3$ is strictly increasing in the interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

Hence, f_3 is neither increasing nor decreasing in interval $\left(0, \frac{\pi}{2}\right)$

D) Let $f_4(x) = \tan x$

$\therefore f_4'(x) = \sec^2 x$

In interval $\left(0, \frac{\pi}{2}\right)$, $f_4'(x) = \sec^2 x > 0$

$\therefore f_4$ is strictly increasing in the interval $\left(0, \frac{\pi}{2}\right)$

Therefore, functions cos x and cos 2x are strictly decreasing in $\left(0, \frac{\pi}{2}\right)$

Hence, the correct options are A and B

13.

On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ decreasing ?

(A) (0,1) **(B) $\frac{\pi}{2}, \pi$** **(C) $0, \frac{\pi}{2}$** **(D) None of these**

Ans - We have, $f(x) = x^{100} + \sin x - 1$

$\therefore f'(x) = 100x^{90} + \cos x$

In interval $(0,1)$, $\cos x > 0$ and $100x^{90} > 0$

$\therefore f'(x) > 0$

Thus, the function f is strictly increasing in the interval $(0,1)$

In interval $\left(\frac{\pi}{2}, \pi\right)$, $\cos x < 0$ and $100\mathbf{x}^{90} > 0$

Also, $100x^{90} > \cos x$

$\therefore f'(x) > 0$ in $\left(\frac{\pi}{2}, \pi\right)$

Thus, the function f is strictly increasing in the interval $\left(\frac{\pi}{2}, \pi\right)$

In the interval $\left(0, \frac{\pi}{2}\right)$, $\cos x < 0$ and $100x^{90} > 0$

$\therefore 100x^{90} + \cos x > 0$

$\Rightarrow f'(x) > 0$ on $\left(0, \frac{\pi}{2}\right)$

$\therefore$ f is strictly increasing in the interval $\left(0, \frac{\pi}{2}\right)$

Hence, the function f is strictly decreasing in none of the intervals. The correct option is D

14.

For what values of a the function f given by $f(x) = x^2 + ax + 1$ is increasing on [1, 2]?

Ans - We have, $f(x) = x^2 + ax + 1$

$\therefore f'(x) = 2x + a$

Now, the function f will be increasing in (1,2) if
$f'(x) > 0$ in $(1, 2)$.

$\Rightarrow 2x + a > 0$

$\Rightarrow 2x > -a$

$\Rightarrow x > \frac{-a}{2}$

As a result, we must determine the smallest value of a such that

$x > \frac{-a}{2}, x \in (1,2)$

$\Rightarrow x > \frac{-a}{2} (1 < x < 2)$

Thus, the least value is given by

$\frac{-a}{2} = 1$

$\Rightarrow a = -2$

Hence, the required value of a is -2

15.

Let I be any interval disjoint from [−1, 1]. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on I

Ans - We have, $f(x) = x + \frac{1}{x}$

$$\therefore f'(x) = 1 - \frac{1}{x^2}$$

Now, $f(x) = 0$

$\Rightarrow x = \pm 1$

Real line is divided into three disjoint intervals i.e. $(-\infty, -1), (-1,1), (1, \infty)$ by the points $x = 1, -1$

In the interval (-1,1) we observe $-1 < x < 1$

$\Rightarrow x^2 < 1$

$\Rightarrow 1 < \frac{1}{x^2}, x \neq 0$

$\therefore f'(x) = 1 - \frac{1}{x^2} < 0$ on $(-1,1) \sim \{0\}$

$\therefore$ f is strictly decreasing on $(-1,1) \sim \{0\}$

In the interval, $(-\infty, -1)$and$(1, \infty)$, it is observed that:

$x < -1$

$\Rightarrow x^2 > 1$

$\Rightarrow 1 > \frac{1}{x^2}$

$\Rightarrow 1 - \frac{1}{x^2} > 0$

$\therefore f'(x) = 1 - \frac{1}{x^2} > 0$ on$(-\infty, -1)$and$(1, \infty)$

$\therefore$ f is strictly increasing on $(-\infty, -1)$and$(1, \infty)$.

Hence, the function f is strictly increasing in the interval I disjoint from (1, 1)

Hence, the given result is proved

16.

Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Ans - We have, $f(x) = \log \sin x$

$$\therefore f'(x) = \frac{1}{\sin x}\cos x = \cot x$$

In interval, $\left(0, \frac{\pi}{2}\right), f'(x) = \cot x > 0$

$\therefore$ f is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

In interval, $\left(\frac{\pi}{2}, \pi\right), f'(x) = \cot x < 0$

$\therefore$ f is strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$

17.

Prove that the function f is given by $f(x) = \log\cos x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$

Ans - We have, $f(x) = \log\cos x$

$\therefore f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$

In interval $\left(0, \frac{\pi}{2}\right)$, $\tan x > 0 = -\tan x < 0$

$\therefore f'(x) < 0$ on $\left(0, \frac{\pi}{2}\right)$

$\therefore$ f is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$

In interval $\left(\frac{\pi}{2}, \pi\right)$

$\tan x < 0 = -\tan x > 0$

$\therefore f'^{(x)} > 0$ on $\left(\frac{\pi}{2}, \pi\right)$

$\therefore$ f is strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$

18.

Prove that the function given by $f(x) = x^3 - 3x^2 + 3x = 100$ is increasing in R.

Ans - We have, $fx = x^3 - 3x^2 + 3x = 100$

$f'x = 3x^2 - 6x + 3$

$= 3(x^2 - 2x + 1)$

$= 3(x - 1)2$

For any $x \in R$

$(x - 1)^2 > 0$

Thus, $f'(x)$ is always positive in R

Hence, the given function f is increasing in R

19.

The interval in which $y = x^2e^{-x}$ is increasing at

(A) $(-\infty, \infty)$ **(B) $(-2, 0)$**

(C) $(2, \infty)$ **(D) $(0, 2)$**

Ans - We have, $y = x^2e^{-x}$

$\therefore \frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$

Now, $\frac{dy}{dx} = 0$

$\Rightarrow x = 0, 2$

The points $x = 0, 2$ divided the real line into the three disjoint intervals i.e., $(-\infty, 0), (0, 2), (2, \infty)$.

In intervals $(-\infty, 0)$ and $(2, \infty), f'(x) < 0$ as e^{-x} is always positive.

$\therefore$ f is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

In interval $(0, 2), f'(x) > 0$

$\therefore$ f is strictly increasing on $(0, 2)$.

Hence f is strictly increasing in the interval $(0, 2)$. Thus, **D** is the correct option.

Exercise 6.3

1.

Find the maximum and minimum values, if any, of the following given by

(i) $f(x) = (2x - 1)^2 + 3$ **(ii) $f(x) = 9x^2 + 12x + 2$**

(iii) $f(x) = -(x - 1)^2 + 10$ **(iv) $g(x) = x^3 + 1$**

Ans – **(i)** The given function is $f(x) = (2x - 1)^2 + 3$.

It can be observed that $(2x - 1)^2 \geq 0$ for every $x \in R$.

$\Rightarrow f(x) = (2x - 1)^2 + 3 \geq 3$ for every $x \in R$.

Minimum value of f is attained when $2x - 1 = 0$

$\Rightarrow x = \frac{1}{2}$

$\Rightarrow$ Minimum value of f,

$$f\left(\frac{1}{2}\right) = \left(2 \cdot \frac{1}{2} - 1\right)^2 + 3 = 3$$

Hence, function f does not have a maximum value.

(ii) Given function, $f(x) = 9x^2 + 12x + 2 = (3x^2 + 2)^2 - 2$.

It can be observed that $(3x^2 + 2)^2 \geq 0$ for every $x \in R$.

$\Rightarrow f(x) = (3x^2 + 2)^2 - 2 \geq -2$ for every $x \in R$.

Minimum value of f is attained when $3x + 2 = 0$

$\Rightarrow x = -\frac{2}{3}$

$\Rightarrow$ Minimum value of f,

$$f\left(-\frac{2}{3}\right) = \left(3\left(\frac{-2}{3}\right) + 2\right)^2 - 2 = -2.$$

Hence, function f does not have a maximum value.

(iii) Given function is $f(x) = -(x-1)^2 + 10$.

It can be observed that $(x-1)^2 \geq 0$ for every $x \in R$.

$\Rightarrow f(x) = -(x-1)^2 + 10 \leq 10$ for every $x \in R$.

Minimum value of f is attained when $(x-1) = 0$

$x - 1 = 0, x = 0$

$\Rightarrow$ Minimum value of f,

$f = f(1) = -(1-1)^2 + 10 = 10$

Hence, function f does not have a maximum value.

(iv) Given function is $g(x) = x^3 + 1$.

Hence, function g neither has a maximum value nor a minimum value.

2.

Find the maximum and minimum values, if any, of the following functions given by

(i) $\mathbf{f(x) = |x+2| - 1}$ **(ii)** $\mathbf{g(x) = -|x+1| + 3}$
(iii) $\mathbf{h(x) = \sin\ 2x + 5}$ **(iv)** $\mathbf{f(x) = |\sin\ 4x + 3|}$
(v) $\mathbf{h(x) = x + 4, x \in (-1, 1)}$

Ans – (i) We know that $|x+2| \geq 0$ for every $x \in R$

$\Rightarrow f(x) = |x+2| - 1 \geq -1$ for every $x \in R$

Minimum value of f is attained when $|x+2| = 0$

$\Rightarrow x = -2$

$\Rightarrow$ Minimum value of f,

$f(-2) = |-2+2| - 1 = -1$

Hence, function f does not have a maximum value.

(ii) We know that $-|x+1| \leq 0$ for every $x \in R$

$\Rightarrow g(x) = -|x+1| + 3 \leq 3$ for every $x \in R$

Minimum value of g is attained when $|x+1| = 0$

$\Rightarrow x = -1$

$\Rightarrow$ Minimum value of g,

$g(-1) = -|-1+1| + 3 = 3$

Hence, function g does not have a maximum value.

(iii) We know that $-1 \leq \sin\ 2x \leq 1$

$-1 + 5 \leq \sin\ 2x + 5 \leq 1 + 5$

$4 \leq \sin\ 2x + 5 \leq 6$

Hence, the maximum and minimum values of h are 6 & 4 respectively.

(iv) We know that $-1 \leq \sin\ 4x \leq 1$

$2 \leq \sin\ 4x + 3 \leq 4$

$2 \leq |\sin\ 4x + 3| \leq 4$

Hence, the maximum and minimum values of f are 4 & 2 respectively.

(v) Here, if a point x_0 is closest to -1 , then we find $\frac{x_0}{2} + 1 < x_0 + 1$ for all $x_0 \in (-1, 1)$.

Also, if x_1 is closet to -1 , then we find $x_1 + 1 < \frac{x_1+1}{2} + 1$ for all $x_0 \in (-1,1)$.

Hence, function h(x) has neither maximum nor minimum value at $(-1,1)$.

3.

Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $\mathbf{f(x) = x^2}$

(ii) $\mathbf{g(x) = x^3 - 3x}$

(iii) $\mathbf{h(x) = \sin\ x + \cos . 0 < x < \frac{\pi}{2}}$

(iv) $\mathbf{f(x) = \sin\ x - \cos\ x, 0 < x < 2\pi}$

(v) $\mathbf{f(x) = x^3 - 6x^2 + 9x + 15}$

(vi) $\mathbf{g(x) = \frac{x}{2} + \frac{2}{x}, x > 0}$

(vii) $\mathbf{g(x) = \frac{1}{x^2+2}}$

(viii) $\mathbf{f(x) = x\sqrt{1 - x}, x > 0}$

Ans – (i) $f(x) = x^2$

$\therefore f'(x) = 2x$

Now, $f'(x) = 0 \Rightarrow x = 0$

Thus, $x = 0$ is the only critical point that could be the point of local maxima or local minima of f.

We have $f''(0) = 2$, which is positive.

Therefore, by second derivative test, $x = 0$ is a point of local minima and local minimum value of f at $x = 0$ is $f(0) = 0$.

(ii) $g(x) = x^3 - 3x$

$\therefore g'(x) = 3x^2 - 3$

Now,
$g'(x) = 0 \Rightarrow 3x^2 = 3 \Rightarrow x = \pm 1$

$g''(x) = 6x$

$g''(1) = 6 > 0$

$g''(-1) = -6 < 0$

By second derivative test, $x = 1$ is a point of local minima and local minimum value of g at $x = 1$ is $g(1) = 1^3 - 3 = -2$.

(iii) $h(x) = \sin\ x + \cos\ .0 < x < \frac{\pi}{2}$

$\therefore h'(x) = \cos\ x + \sin\ x$

$h'(x) = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1$
$\Rightarrow x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

$h''(x) = -\sin x - \cos x$

$h''\left(\frac{\pi}{4}\right) = -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0$

Therefore, by second derivative test, $x = \frac{\pi}{4}$ is a point of local maxima and the local maximum value of h at $x = \frac{\pi}{4}$

$h\left(\frac{\pi}{4}\right)^4 = \sin\ \frac{\pi}{4} + \cos\ \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

(iv) $f(x) = \sin x - \cos x, 0 < x < 2\pi$

$\therefore f'(x) = \cos x + \sin x$

$f'(x) = 0 \Rightarrow \cos x = -\sin x$

$\tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0,2\pi)$

$f''(x) = -\sin x + \cos x$

$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\sqrt{2} < 0$

$f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = \sqrt{2} > 0$

Therefore, by second derivative test, $x = \frac{3\pi}{4}$ is appoint of local maxima and the local maximum value of f at $x = \frac{3\pi}{4}$ is $f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4}\cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$

However, $x = \frac{7\pi}{4}$ is a point of local minima and the local minimum value of f at $x = \frac{7\pi}{4}$ is

$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}.$

(v) Here, if a point x_0 is closest to -1 , then we find $\frac{x_0}{2} + 1 < x_0 + 1$ for all $x_0 \in (-1, 1)$.

Also, if x_1 is closet to -1 , then we find $x_1 + 1 < \frac{x_1+1}{2} + 1$ for all $x_0 \in (-1,1)$.

Hence, function h(x) has neither maximum nor minimum value at $(-1,1)$.

(vi) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

$\therefore g'(x) = \frac{1}{2} - \frac{2}{x^2}$

Now, $g'(x) = 0$ gives $\frac{2}{x^2} = \frac{1}{2}$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Since $x > 0$ we take $x = 2$.

Now, $g''(x) = \frac{4}{x^3}$

$$g''(2) = \frac{\frac{4}{2^3}}{2''} = \frac{1}{2} > 0$$

Therefore, by second derivative test, $x = 2$ is a point of local minima and the local minimum value of g at $x = 2$ is $g(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2$.

(vii) $g(x) = \frac{1}{x^2+2}$

$\therefore g'(x) = \frac{-(2x)}{(x^3 + 2)^2}$

$g'(x) = 0 \Rightarrow \frac{-(2x)}{(x^3 + 2)^2} = 0 \Rightarrow x = 0$

Now, for values close to $x = 0$ and to the left of $0, g'(x) > 0$.

Also, for values close to $x = 0$ and to the right of $g'(x) < 0$.

Therefore, by first derivative test $x = 0$ is a point of local maxima and the local maximum value of $g(0)$is $\frac{1}{0+2} = \frac{1}{2}$.

(viii) $f(x) = x\sqrt{1-x}, x > 0$

$$\therefore f'(x) = x\sqrt{1-x} + x \cdot \frac{1}{2\sqrt{1-x}}(-1) = \sqrt{1-x} - \frac{x}{2\sqrt{1-x}}$$

$$= \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

$$f'(x) = 0 \Rightarrow \frac{2-3x}{2\sqrt{1-x}} = 0$$

$$\Rightarrow 2-3x = 0 \Rightarrow x = \frac{2}{3}$$

$$f''(x) = \frac{1}{2}\left[\frac{\sqrt{1-x}(-3) - (2-3x)\left(\frac{-1}{2\sqrt{1-x}}\right)}{1-x}\right]$$

$$= \frac{\sqrt{1-x}(-3) + 2(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)}{2(1-x)}$$

$$= \frac{3x-4}{4(1-x)^{\frac{3}{2}}}$$

$$f'\left(\frac{2}{3}\right) = \frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0$$

Therefore, by second derivative test, $x = \frac{2}{3}$ is a point of local maxima and the local maximum value of f at $x = \frac{2}{3}$ is $f\left(\frac{2}{3}\right) = \frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}} = \frac{-1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}} < 0.$

4.

Prove that the following functions do not have maxima or minima:

(i) $\mathbf{f(x) = e^x}$

(ii) $\mathbf{g(x) = \log\, x}$

(iii) $\mathbf{h(x) = x^3 + x^2 + x + 1}$

Ans – **(i)** $f(x) = e^x$

$\therefore f'(x) = e^x$

Now, if $f'(x) = 0$ then $e^x = 0$. But the exponential function can never assume 0 for any value of **x**.

$\therefore$ There does not exist $c \in R$ such that $f'(c) = 0$.
Hence, function f does not have maxima or minima.

(ii) We have, $g(x) = \log\, x$

$$\therefore g'(x) = \frac{1}{x}$$

Since $\log\, x$ is defined for a positive number x, $g'(x) > 0$ for any x

$\therefore$ There does not exist $c \in R$ such that $g'(c) = 0$.
Hence, function g does not have maxima or minima.

(iii) We have, $h(x) = x^3 + x^2 + x + 1$

$\therefore h'(x) = 3x^2 + 2x + 1$

$\therefore$ There does not exist $c \in R$ such that $h'(c) = 0$.
Hence, function h does not have maxima or minima.

5.

Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $\mathbf{f(x) = x^3, x \in [-2, 2]}$

(ii) $\mathbf{f(x) = \sin x + \cos x, x \in [0, \pi]}$

(iii) $\mathbf{f(x) = 4x - \frac{1}{2}x^2, x \in \left[-2, \frac{9}{2}\right]}$

(iv) $\mathbf{f(x) = (x - 1)^2 + 3, x \in [-3, 1]}$

Ans – **(i)** Given function is $f(x) = x^3$.

$\therefore f'(x) = 3x^2$

Now, $f'(x) = 0 \Rightarrow x = 0$

Then, we evaluate the value of f at critical point $x = 0$ and at endpoints of the interval $[-2,2]$.

$f(0) = 0$

$f(-2) = (-2)^3 = -8$

$f(2) = (2)^3 = 8$

Hence, we can conclude that the absolute maximum value of f on $[-2,2]$ is 8 occurring at $x = -2$. Also, the absolute minimum value of f on $[-2,2]$ is -8 occurring at $x = -2$.

(ii) Given function is $f(x) = \sin x + \cos x$

$\therefore f'(x) = \cos x - \sin x$

Now, $f'(x) = 0 \Rightarrow \sin x = \cos x$

$\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$

Then, we evaluate the value of f at critical point $x = \frac{\pi}{4}$ and at the endpoints of the interval $[0, \pi]$.

$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

$f(0) = \sin 0 + \cos 0 = 1$

$f(\pi) = \sin \pi + \cos \pi = 1$

Hence, we can conclude that the absolute maximum value of f on $[0, \pi]$ is $\sqrt{2}$ occurring at $x = \frac{\pi}{4}$. Also, the absolute minimum value of f on $[0, \pi]$ is -1 occurring at $x = \pi$.

(iii) Given function is $f(x) = 4x - \frac{1}{2}x^2$

$$\therefore f'(x) = 4x - \frac{1}{2}x^2$$

Now, $f'(x) = 0 \Rightarrow x = 4$

Then, we evaluate the value of f at critical point $x = 4$ and at the endpoints of the interval $\left[-2, \frac{9}{2}\right]$.

$$f(4) = 16 - \frac{1}{2}(16) = 8$$

$$f(-2) = -8 - \frac{1}{2}(4) = -10$$

$$f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = 18 - \frac{81}{8} = 7.875$$

Hence, we can conclude that the absolute maximum value of f on $\left[-2, \frac{9}{2}\right]$ is 8 occurring at $x = 4$.

Also, the absolute minimum value of f on $\left[-2, \frac{9}{2}\right]$ is -10 occurring at $x = -2$.

(iv) Given function is $f(x) = (x - 1)^2 + 3$

$\therefore f'(x) = 2(x - 1)$

Now, $f'(x) = 0 \Rightarrow 2(x - 1) = 0, x = 1$

Then, we evaluate the value of x at critical point $x = 1$ and at the endpoints of the interval $[-3,1]$.

$f(1) = (1 - 1)^2 + 3 = 3$

$f(-3) = (-3 - 1)^2 + 3 = 19$

Hence, we can conclude that absolute maximum value of f on $[-3,1]$ is 19 taking place at $x = -3$. Also, absolute minimum value of f on $[-3,1]$ is 3 taking place at $x = 1$.

6.

Find the maximum profit that a company can make if the profit function is given by $p(x) = 41 - 24x - 18x^2$.

Ans - Profit function is given as $p(x) = 41 - 24x - 18x^2$.

$\therefore p'(x) = -24 - 36x$

$p''(x) = -36$

Now, $p'(x) = 0$

$\Rightarrow x = \frac{-24}{36} = -\frac{2}{3}$

Also,

$p''\left(\frac{-2}{3}\right) = -36 < 0$

By second derivatives test, $x = -\frac{2}{3}$ is the point of local maximum of p. Therefore,

Maximum profit $= p\left(-\frac{2}{3}\right) = 41 - 24\left(-\frac{2}{3}\right) - 18\left(-\frac{2}{3}\right)^2$

$= 41 + 16 - 8 = 49$

Hence, the maximum profit that the company can make is 49 units.

7.

Find both the maximum value and the minimum value of $3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$.

Ans - Let's take $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$ on 0,3

$f'(x) = 12x^3 - 24x^2 + 24x - 48$

Now $f'(x) = 0$

$\Rightarrow 12x^3 - 24x^2 + 24x - 48 = 0$

$\Rightarrow x^3 - 2x^2 + 2x - 4 = 0$

$\Rightarrow (x - 2)(x^2 + 2) = 0$

$\Rightarrow x = 2$ or $x = \pm\sqrt{2}$

As $x = \pm\sqrt{2}$ is imaginary, therefore this value is not considered.

At $x = 2, f(2) = 3(16) - 8(8) + 12(4) - 48(2) + 25 = -39$

At $x = 0, f(0) = 3(0) - 8(0) + 12(0) - 48(0) + 25 = 25$

At $x = 3, f(3) = 3(81) - 8(27) + 12(9) - 48(3) + 25 = 16$

Therefore, the minimum value of the given function is -39 and maximum value is 25 .

8.

At what points in the interval $[0, 2\pi]$ does the function $\sin\ 2x$ attain, its maximum value?

Ans – Let $f(x) = \sin\ 2x$

$\therefore f'(x) = 2\cos 2x$

Now, $f'(x) = 0 \Rightarrow \cos\ 2x = 0$

$\Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Now, we evaluate values of f at critical points $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and at the endpoints of interval $[0,2\pi]$.

$f\left(\frac{\pi}{4}\right) = \sin\ \frac{\pi}{2} = 1,$

$f\left(\frac{3\pi}{4}\right) = \sin\ \frac{3\pi}{2} = -1,$

$f\left(\frac{5\pi}{4}\right) = \sin\ \frac{5\pi}{2} = 1,$

$f\left(\frac{7\pi}{4}\right) = \sin\ \frac{7\pi}{2} = -1,$

$f(0) = \sin\ 0 = 0,$

$f(2\pi) = \sin\ 2\pi = 0$

Hence, we can conclude that the absolute maximum value of $f[0,2\pi]$ is occurring at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

9.

What is the maximum value of the function $\sin x + \cos x$?

Ans - Let $f(x) = \sin\ x + \cos\ x$

$\therefore f'(x) = \cos\ x - \sin\ x$

$f'(x) = 0 \Rightarrow \sin x = \cos x$

$\Rightarrow \tan\ x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$

$f'(x) = -\sin\ x - \cos\ x = -(\sin\ x + \cos\ x)$

Now, when $(\sin\ x + \cos\ x)$ is positive, i.e., when $\sin\ x$ and $\cos\ x$ are both positive, $f'(x)$ will be negative.

We also know that in first quadrant, both $\sin\ x$ and $\cos\ x$ are positive. Then, $f''(x)$ will be negative when $x \in \left(0, \frac{\pi}{2}\right)$.

As a result, we consider $x = \frac{\pi}{4}$

$$f''\left(\frac{\pi}{4}\right) = -\left(\sin\ \frac{\pi}{4} + \cos\ \frac{\pi}{4}\right) = -\left(\frac{2}{\sqrt{2}}\right) = -\sqrt{2} < 0$$

By second derivative test, f will be maximum at $x = \frac{\pi}{4}$ and maximum value of f is

$$f\left(\frac{\pi}{4}\right) = \sin\ \frac{\pi}{4} + \cos\ \frac{\pi}{4} = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

10.

Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.

Ans - Let $f(x) = 2x^3 - 24x - 107$

$\therefore f'(x) = 6x^2 - 24 = 6(x^2 - 4)$

Now, $f'(x) = 0 \Rightarrow 6(x^2 - 4) = 0$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

We first consider the interval [1,3].

Then, we evaluate the value of f at the critical point $x \in [1,3]$ and at the endpoints of the interval [1,3]

$f(2) = 2(8) - 24(2) + 107 = 75$

$f(1) = 2(1) - 24(1) + 107 = 85$

$f(3) = 2(27) - 24(3) + 107 = 89$

Hence, the absolute maximum value of f(x) in the interval [1,3] is 89 occurring at $x = 3$

Next, we consider the interval $[-3, -1]$.

Evaluate the value of f at the critical point $x \in [1,3]$ and at the endpoints of the interval $[-3, -1]$.

$f(-3) = 2(-27) - 24(-3) + 107 = 125$

$f(-1) = 2(-1) - 24(-1) + 107 = 129$

$f(-2) = 2(-8) - 24(-2) + 107 = 139$

Hence, the absolute maximum value of f(x) in the interval $[-3, -1]$ is 139 occurring at $x = -2$.

11.

It is given that at $x = 1$ the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$. Find the value of a.

Ans - Let $f(x) = x^4 - 62x^2 + ax + 9$

$\therefore f'(x) = 4x^2 - 124x + a$

It is given that function f attains its maximum value on the interval $[0, 2]$ at $x = 1$.

$\therefore f'(1) = 0$

$\Rightarrow 4 - 124 + a = 0$

$\Rightarrow a = 120$

Hence, the value of a is 120 .

12.

Find the maximum and minimum values of $x + \sin\ 2x$ on $[0, 2\pi]$.

Ans - Let $f(x) = x + \sin\ 2x$

$\therefore f'(x) = 1 + 2\cos\ 2x$

Now, $f'(x) = 0 \Rightarrow \cos\ 2x = -\frac{1}{2} = \cos\ \frac{2\pi}{3}$

$2x = 2\pi \pm \frac{2\pi}{3}, n \in Z$

$\Rightarrow x = \pi \pm \frac{\pi}{3}, n \in Z$

$\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \in [0,2\pi]$

Then, we evaluate the value of at f critical points $\Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ and the end points of the interval $[0,2\pi]$.

$$f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

$$f\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + \sin\frac{8\pi}{3} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

$$f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sin\frac{10\pi}{3} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$

$$f(0) = 0 + \sin 0 = 0$$

$$f(2\pi) = 2\pi + \sin 4\pi = 2\pi$$

Hence, we can conclude that the absolute maximum value of $f(x)$ in the interval $[0,2\pi]$ is 2π occurring at $x = 2\pi$ and the absolute maximum value of $f(x)$ in the interval $[0, 2\pi]$ is 0 occurring at $x = 0$.

13.

Find two numbers whose sum is 24 and whose product is as large as possible.

Ans - Let one number be x. Then, other number is $(24 - x)$.

Let $p(x)$ denote the product of the two numbers.

$\Rightarrow P(x) = x(24 - x) = 24x - x^2$

$\therefore P'(x) = 24 - 2x$

$P''(x) = -2$

Now, $P'(x) = 0 \Rightarrow x = 12$

Also, $P''(12) = -2 < 0$

By second derivative test, $x = 12$ is the point of local maxima of P .

Hence, the product of the numbers is the maximum when the numbers are 12 and $24 - 12 = 12$.

14.

Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

Ans – Let us assue two numbers be x and y then $x + y = 60$.

$\Rightarrow y = 60 - x$

Let $f(x) = xy^3$

$\Rightarrow f(x) = x(60 - x)^3$

$\therefore f'(x) = (60 + x)^3 - 3x(60 - x)^2 = (60 + x)^3[60 - 4x]$

And, $f''(x) = -2(60 - x)(60 - 4x) - 4(60 - x)^2$

$= -2(60 - x)(180 - 6x)$

$= -12(60 - x)(30 - x)$

Now, $f'(x) = 0$

$\Rightarrow x = 60, 15$

When $x = 60, f''(x) = 0$

When $x = 15, f''(x) = -12(60 - 15)(30 - 15) < 0$.

By second derivative test, $x = 15$ is a point of local maxima of f. Thus, function xy^3 is maximum when $x = 15$ and $y = 60 - 15 = 45$

Hence, the required numbers are 15 and 45 .

15.

Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

Ans - Let us assue two numbers be x and y then $x + y = 35$.

$\Rightarrow y = (35 - x)$

Let $p(x) = x^2y^5$.

$\Rightarrow P(x) = x^2(35 - x)^5$

$\therefore P'(x) = 2x(35 - x)^5 - 5x^2(35 - x)^4$

$= x(35 - x)^4[2(35 - x) - 5x]$

$= 7x(35 - x)^4(10 - x)$

And,

$$P''(x) = 7(35 - x)^4(10 - x) + 7x[-(35 - 5)^4 - 4(35 - x)^3(10 - x)]$$

$$= 7(35 - x)^4(10 - x) - 7x(35 - x)^4 - 28x(35 - x)^3(10 - x)$$

$= 7(35 - x)^3[350 - 45x + x^2 - 35x + x^2 - 40x + 4x^2]$

$= 7(35 - x)^3(6x^2 - 120x + 350)$

Now, $P'(x) = 0$

$\Rightarrow x = 0, x = 35, x = 10$

When, $x = 35, f'(x) = f(x) = 0$

$y = 35 - 35 = 0.$

$\Rightarrow$ Product x^2y^5 will be equal to 0.

When $x = 0, y = 35 - 0 = 35$

$\Rightarrow$ Product x^2y^5 will be equal to 0.

$x = 0$ and $x = 35$ cannot be the possible values of x .

When $x = 10$ we have:

$P''(x) = 7(35 - 10)^3(-250) < 0$

By second derivative test, $P(x)$ will be the maximum when
$x = 10$ and $y = 35 - 10 = 25$

Hence, the required numbers are 10 and 25 .

16.

Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

Ans - Let one number be x. Then, other number is $(16 - x)$.

Let sum of the cubes of these numbers be denoted by $S(x)$.

$\Rightarrow S(x) = x^3 + (16 - x)^3$

$\therefore S'(x) = 3x^2 - 3(16 - x)^2,$

$S''(x) = 6x + 6(16 - x)$

Now, $S'(x) = 0 \Rightarrow 3x^2 - 3(16 - x)^2 = 0$

$$\Rightarrow x^2 - (16 - x)^2 = 0 \Rightarrow x = \frac{256}{32} = 8$$

Now, $S''(8) = 6(8) + 6(16 - 8) = 96 > 0$

By second derivative test, $x = 8$ is the point of local minima of S. Hence, sum of the cubes of the numbers is minimum when the numbers are 8 and $16 - 8 = 8$.

17.

A square piece of tin od side 18 cm is to be made into a box without a top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Ans: Let side of the square be cut off by x cm. Then, length and the breath of box will be (18 – 2x) cm each.

Box has a height of x cm. As a result, the box's volume V(x) is given by, $V(x) = x(18-2x)^2$

$\therefore V'(x) = (18-2x)^2 - 4x(18-2x) = (18-2x)[18-2x-4x]$

$=12(9-x)(3-x)$

And, $V''(x)=12[-(9-x)-(3-x)] = -24(6-x)$

Now, $v'(x) = 0 \Rightarrow x = 9, 3$

If $x = 9$, then length and breadth will become 0.

$\therefore x = 9 \;\; \Rightarrow x = 3$

$\Rightarrow V''(3) = -24(6-3) = -72 < 0$

$\therefore$ By second derivative test, $x = 3$ is the point of maxima of V.

As a result, volume of box obtained is the biggest achievable by removing a square of side 3 cm from each corner of square tin and building a box from remaining sheet.

18.

A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without a top, by cutting off squares from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Ans: Let side of the square be cut by x cm. Then, height of the box is x, length is 45−2x , and breadth is 24−2x.

$V(x) = x(45 - 2x)(24 - 2x) = 4x^3 - 138x^2 + 1080x$

$V'(x) = 12x^2 - 276 + 1080 = 12(x^2 - 23x + 90)$

$= 12(x - 18)(x - 5)$

$V''(x) = 24x - 276 = 12(2x - 23)$

Now, $V'(x) = 0 \Rightarrow x = 18, 5$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet,

Thus, x cannot be equal to 18.

$x=5$

Now, $V''(5) = 12(10 - 23) = -156 < 0$

Therefore, by second derivative test $x = 5$ is the point of maxima. Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5cm.

19.

Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Ans - By Pythagoras theorem, we have:

$(2a)^2 = l^2 + b^2$

$\Rightarrow b^2 = 4a^2 - l^2$

$\Rightarrow b = \sqrt{4a^2 - l^2}$

Area of the rectangle, $A = l\sqrt{4a^2 - l^2}$

$$\therefore \frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l)$$

$$= \sqrt{4a^2 - l^2} + \frac{l^2}{\sqrt{4a^2 - l^2}} = \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$$

$$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2)\frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{(4a^2 - l^2)}$$

$$= \frac{(4a^2 - l^2)(-4l) + 1(4a^2 - 2l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$= \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

Now, $\frac{dA}{dl} = 0$ gives $4a^2 = 2l^2 \Rightarrow l = \sqrt{2}a$

$\Rightarrow b = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2} = \sqrt{2}a$

When $l = \sqrt{2}a$

$$\frac{d^2A}{dl^2} = \frac{-2(\sqrt{2}a)(-6a^2 - 2a^2)}{2\sqrt{2}a^3} = \frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3} = -4 < 0$$

By second derivative test, when $l = \sqrt{2}a$ then area of the rectangle is the maximum. Since $l = b = \sqrt{2}a$ the rectangle is a square.

Hence, it has been proved that the square has the maximum area of all the rectangles inscribed in the given fixed circle.

20.

Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Ans - Let r and h denote radius and height of cylinder respectively. Then, cylinder will have Surface Area (S):

$$S = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow h = \frac{S - 2\pi r^2}{2\pi r}$$

Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2 \left[\frac{S - 2\pi r^2}{2\pi r}\right] = \frac{r}{2}(S - 2\pi r^2) = \frac{1}{2}(Sr - 2\pi r^3)$$

$$\frac{dV}{dr} = \frac{S}{2} - \frac{6\pi r^2}{2} = \frac{S}{2} - 3\pi r^2$$

$$\frac{d^2V}{dr^2} = -6\pi r$$

Now, $\frac{dV}{dr} = 0$

$$\Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$$

When $r^2 = \frac{S}{6\pi}$ then,

$$h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r$$

Hence, the volume is the maximum when the height is twice the radius i.e. when the height is equal to the diameter

21.

Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

Ans - Let r and h be radius and height of cylinder respectively.

Volume (V) of cylinder will be,

$$V = \pi r^2 h = 100$$

$$\Rightarrow h = \frac{100}{\pi r^2}$$

Surface area (S) of cylinder will be,

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\left(\frac{100}{\pi r^2}\right)$$

$$S = 2\pi r^2 + \frac{200}{r} = 2\pi r^2 + 200r^{-1}$$

$$\frac{dS}{dr} = 4\pi r + (-200r^{-2}) = 4\pi r - \frac{200}{r^2}$$

Now, $\frac{dS}{dr} = 0$

$\Rightarrow 4\pi r = \frac{200}{r^2}$

$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$

$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

When $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}, \frac{d^2S}{dr^2} > 0$

By second derivative test, when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm, the surface area is the minimum.

When $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}, h = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Hence the required dimensions of the can which has the minimum surface area are given as radius $= \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height $= 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

22.

A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Ans - Let a piece of length I be cut from given wire to make a square. Then, the other piece of wire to be made into a circle is of length $(28 - l)m$.

Let r be the circle's radius.

$2\pi r = 28 - l$

$\Rightarrow r = \frac{1}{2\pi}(28 - l).$

The combined areas of the square and the circle (A) is given by,

$$A = l^2 + \pi r^2 = \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28 - l)\right]^2$$

$$= \frac{l^2}{16} + \frac{1}{4\pi}(28 - l)^2$$

$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28 - l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28 - l)$$

$$\frac{d^2A}{dl^2} = \frac{l}{8} + \frac{1}{2\pi} > 0$$

Now, $\frac{dA}{dl} = 0$

$$\Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28 - l) = 0$$

$$\Rightarrow (\pi + 4)l - 112 = 0$$

$$\Rightarrow l = \frac{112}{\pi + 4}$$

$\Rightarrow$ When $l = \frac{112}{\pi + 4}, \frac{d^2A}{dl^2} > 0$

By second derivative test, area is minimum when $l = \frac{112}{\pi+4}$.

Hence, when length of the wire is $\frac{112}{\pi+4}$ cm in making the square the combined area is the minimum while the length of the wire in making the circle is $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$ cm.

23.

Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

Ans - Let r and h be the radius and height of the cone respectively inscribed in a sphere of radius R.

Let V be the volume of the cone.

$$V = \frac{1}{3}\pi r^2 h$$

Height of the cone is given by,

$$h = R + AB = R + \sqrt{R^2 - r^2} \text{ (ABC is a right triangle)}$$

$$V = \frac{1}{3}\pi r^2 \left(R + \sqrt{R^2 - r^2}\right)$$

$$\Rightarrow V = \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2 \sqrt{R^2 - r^2}$$

$$\frac{dV}{dr} = \frac{2}{3}\pi r R + \frac{2}{3}\pi r \sqrt{R^2 - r^2} + \frac{1}{3}\pi r^2 \cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= \frac{2}{3}\pi r R + \frac{2}{3}\pi r \sqrt{R^2 - r^2} - \frac{1}{3}\pi \cdot \frac{r^3}{\sqrt{R^2 - r^2}}$$

$$\Rightarrow \frac{dV}{dr} = \frac{2}{3}\pi r R + \frac{2\pi R^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$$

$$\frac{d^2V}{dr^2} = \frac{2}{3}\pi r R + \frac{9(R^2 - r^2)(2\pi R^2 - 9\pi r^2) + 2\pi r R^2 - 3\pi r^4}{27(R^2 - r^2)^{\frac{3}{2}}}$$

Now, $\frac{dV}{dr} = 0$

$$\Rightarrow \pi \frac{2}{3} rR = \frac{3\pi r^3 - 2\pi R^2}{3\sqrt{R^2 - r^2}}$$

$$\Rightarrow 2R = 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$$

$$\Rightarrow 4R^4 - 4R^2r^2 = 9r^4 + 4R$$

$$\Rightarrow 9r^4 = 8R^2r^2$$

$$\Rightarrow r^2 = \frac{8}{9}R^2$$

When $r^2 = \frac{8}{9}R^2$ then $\frac{d^2V}{dr^2} < 0$

By second derivative test, the volume of the cone is the maximum when $r^2 = \frac{8}{9}R^2$.

When $r^2 = \frac{8}{9}R^2$,

$$h = R + \sqrt{R^2 - \frac{8}{9}R^2} = R + \frac{R}{3} = \frac{4}{3}R$$

$$\therefore V = \frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{8}{27} \times (\text{ Volume of sphere })$$

Hence, the volume of the largest cone that can be inscribed in the sphere is $\frac{8}{27}$ the volume of the sphere.

24.

Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Ans - Let r and h be the radius and the altitude of the cone respectively.

Volume (V) of cone will be,

$$V = \frac{1}{3\pi}\pi r^2 h \Rightarrow h = \frac{3V}{r^2}$$

The cone has the following surface area (S),

$S = \pi rl$ (I = slant height)

$= \pi r\sqrt{r^2 + h^2}$

$$= \pi r\sqrt{r^2 + \sqrt{\frac{9V^2}{\pi^2 r^4}}} = \pi\frac{r\sqrt{9^2 r^2 + V^2}}{\pi r^2}$$

$$= \frac{1}{r}\sqrt{\pi^2 r^6 + 9V^2}$$

$$\frac{dS}{dr} = \frac{r \cdot \frac{6_6 r^5}{2\sqrt{\pi^2 r^6 + 9V^2}} - \sqrt{\pi^2 r^6 + 9V^2}}{r^2}$$

$$= \frac{2\pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}}$$

Now, $\frac{dS}{dr} = 0 \Rightarrow 2\pi^2 r^6 = 9V^2 \Rightarrow r^6 = \frac{9V^2}{2\pi^2}$

Thus, it can be easily verified that when $r^6 = \frac{9V^2}{2\pi^2}, \frac{d^2S}{dr^2} < 0$

By second derivative test, the surface area of the cone is the least when $r^6 = \frac{9V^2}{2\pi^2}$

When, $r^6 = \frac{9V^2}{2\pi^2}, h = \frac{3V}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}} = \frac{3}{\pi r^2}\frac{\sqrt{2}\pi r^3}{3} = \sqrt{2}r$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base.

25.

Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

Ans: Let θ be semi-vertical angle of the cone. It is clear that θ∈[0,π2]. Also let r, h, and l be the radius, height, and the slant height of the cone respectively.

Slant height of the cone is given as constant.

Now $r = l\sin\theta$ and $h = l\cos\theta$

Volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(l^2\sin^2\ \theta)(l\cos\ \theta)$$

$$\Rightarrow V = \frac{1}{3}\pi l^3\sin^2\ \theta\cos\ \theta$$

$$\therefore \frac{dV}{d\theta} = \frac{l^3\pi}{3}[\sin^2\ \theta(-\sin\ \theta) + \cos\ \theta(2\sin\ \theta\cos\ \theta)]$$

$$= \frac{l^3\pi}{3}[-\sin^3\ \theta + 2\sin\ \theta\cos^2\ \theta]$$

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}[-3\sin^2\ \theta\cos\ \theta + 2\cos^3\ \theta - 4\sin^2\ \theta\cos\ \theta]$$

$$= \frac{l^3\pi}{3}[2\cos^2\ \theta - 7\sin^2\ \theta\cos\ \theta]$$

Now, $\frac{dV}{d\theta} = 0$

$\Rightarrow \sin^3\ \theta = 2\sin\ \theta\cos^2\ \theta$

$\Rightarrow \tan\ \theta = 2$

$\Rightarrow \theta = \tan^{-1}\ \sqrt{2}$

Now, when $\theta = \tan^{-1}\sqrt{2}$, then

$\tan^2\ \theta = 2$ or $\sin^2\ \theta = 2\cos^2\ \theta$

$$\Rightarrow \frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}[2\cos^3\theta - 14\cos^3\theta]$$

$= -4\pi l^3\cos^3\theta < 0$ (For $\theta \in \left[0, \frac{\pi}{2}\right]$)

By second derivative test, volume (V) is maximum when $\theta = \tan^{-1}\ \sqrt{2}$. Hence, semi-vertical angle of the cone of maximum volume is $\tan^{-1}\ \sqrt{2}$ for a given slant height.

26.

Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$

Ans - Let's assume x be radius, y be height and Θ be semi-vertical angle of the cone.

Total Surface area of cone, $S = \pi x\sqrt{x^2 + y^2} + \pi x^2$

$$x\sqrt{x^2 + y^2} + x^2 = \frac{S}{\pi}$$

Let's say $\frac{S}{\pi} = k$

$$\Rightarrow x\sqrt{x^2 + y^2} = k - x^2$$

Squaring on both sides

$$x^2(x^2 + y^2) = (k - x^2)^2$$

$$\Rightarrow x^2y^2 = k^2 - 2kx^2$$

$$\Rightarrow x^2 = \frac{k^2}{y^2 + 2k} \quad \dots\dots(1)$$

Volume of cone, $V = \frac{1}{3}\pi x^2 y$

$$\Rightarrow V = \frac{1}{3}\pi\left(\frac{k^2}{y^2 + 2k}\right)y = \frac{1}{3}\pi k^2\left(\frac{y}{y^2 + 2k}\right)$$

Differentiate on both sides,

$$\frac{dV}{dy} = \frac{1}{3}\pi k^2 \frac{d}{dy}\frac{y}{y^2 + 2k}$$

$$\Rightarrow \frac{dV}{dy} = \frac{1}{3}\pi k^2\left[\frac{(y^2 + 2k)\cdot 1 - y\cdot 2y}{(y^2 + 2k)^2}\right] \quad \text{(Using quotient rule)}$$

$$\Rightarrow \frac{dV}{dy} = \frac{1}{3}\pi k^2 \frac{(2k - y^2)}{(y^2 + 2k)^2} \quad \dots\dots (2)$$

Now $\frac{dV}{dy} = 0$

$$\frac{1}{3}\pi k^2 \frac{(2k - y^2)}{(y^2 + 2k)^2} = 0$$

$$2k - y^2 = 0$$

$$y^2 = 2k$$

$$y = \pm\sqrt{2k}$$

Height can't be negative so we neglect $y = -\sqrt{2k}$

So, $y = \sqrt{2k}$

As, $\frac{dV}{dy} > 0$, therefore, Volume is maximum at $y = \sqrt{2k}$

From equation (1), We get,

$$x^2 = \frac{k^2}{2k + 2k} = \frac{k^2}{4k} = \frac{k}{4}$$

$$\Rightarrow x = \frac{\sqrt{k}}{2}$$

Now, Semi-vertical angle of the cone,

$$\sin\ \theta = \frac{x}{\sqrt{x^2 + y^2}}$$

$$= \frac{\frac{\sqrt{k}}{2}}{\sqrt{\frac{k}{4} + 2k}} = \frac{\sqrt{k}}{2} \times \sqrt{\frac{4}{9k}} = \frac{1}{3}$$

$$\Rightarrow \theta = \sin^{-1}\frac{1}{3}$$

27.

The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2},4)$ (B) $(2\sqrt{2},0)$ (C) (0, 0) (D) (2, 2)

Ans - Given curve is $x^2 = 2y$

For each value of x, y will be $\frac{x^2}{2}$ and the position of the point will be $\left(x, \frac{x^2}{2}\right)$

The distance d(x) between the points $\left(x, \frac{x^2}{2}\right)$ and (0,5) is given by

$$d(x) = \sqrt{(x-0)^2 + \left(\frac{x^2}{2} - 5\right)^2} = \sqrt{\frac{x^4}{4} - 4x^2 + 25}$$

$$\therefore d'(x) = \frac{(x^3 - 8x)}{\sqrt{x^4 - 16x^2 + 100}}$$

Now, $d'(x) = 0 \Rightarrow x^3 - 8x = 0$

$\Rightarrow x = 0, \pm 2\sqrt{2}$

And $d''(x) = \dfrac{(x^4 - 16x^2 + 100)(3x^2 - 8) - 2(x^3 - 8x)^2}{(x^4 - 16x^2 + 100)^{\frac{3}{2}}}$

When $x = 0$ then $d''(x) = \frac{36(-8)}{6^3} < 0$

When $x = \pm 2\sqrt{2}$ then $d(x) > 0$

By second derivative test, d(x) is the minimum at $x = \pm 2\sqrt{2}$.

When $x = \pm 2\sqrt{2}, y = 4$.

Hence, the point on the curve $x^2 = 2y$ which is nearest to the point (0,5) is $(\pm 2\sqrt{2}, 4)$.

Therefore, the correct option is A.

28.

For all real values of x, the minimum value of

$$\mathbf{f(x) = \frac{1 - x + x^2}{1 + x + x^2} is}$$

(A) 0 **(B) 1** **(C) 3** **(D)** $\mathbf{\frac{1}{3}}$

Ans - Let $f(x) = \frac{1-x+x^2}{1+x+x^2}$

$$f'(x) = \frac{(1+x+x^2)(-1-2x) - (1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

$$= \frac{2(x^2-1)}{(1+x+x^2)^2}$$

$$\therefore f'(x) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Now, $f''(x) = 4\frac{(1+3x-x^3)}{(1+x+x^2)^3}$

And, $f''(1) = \frac{4(1+3-1)}{(1+1+1)^3} = \frac{4}{9} > 0$

And, $f''(-1) = \frac{4(1-3+1)}{(1+1+1)^3} = \frac{-4}{27} < 0$

By second derivative test, f is the minimum at $x = 1$ and the minimum value is given by $f(1) = \frac{1}{3}$

The correct option is D .

29.

The maximum value of $\mathbf{[x(x+1)+1]^{\frac{1}{3}}, 0 \leq x \leq 1}$ **is**

(A) $\mathbf{\left(\frac{1}{3}\right)^{\frac{1}{3}}}$ **(B)** $\mathbf{\frac{1}{2}}$ **(C) 1** **(D) 0**

Ans - Let $f(x) = [x(x+1)+1]^{\frac{1}{3}}$.

$$\therefore f'(x) = \frac{2x+1}{3[x(x+1)+1]^{\frac{2}{3}}}$$

Now, $f(x) = 0 \Rightarrow x = -\frac{1}{2}$

But, $x = -\frac{1}{2}$ is not part of the interval 0,1

Now, evaluate value of f at the endpoints of the interval 0,1 {i.e., at $x = 0$ and $x = 1$}.

$$f(0) = [0(0-1)+1]^{\frac{1}{3}} = 1$$

$$f(0) = [1(1-1)+1]^{\frac{1}{3}} = 1$$

Hence, we can conclude that the maximum value of f is 1 in the interval 0,1.

The correct option is C .

Miscellaneous Exercise

1.

Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

Ans - $f(x) = \frac{\log x}{x}$

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

$$f'(x) = 0$$

$$\Rightarrow 1 - \log x = 0$$

$$\Rightarrow \log x = 1$$

$$\Rightarrow \log x = \log e$$

$$\Rightarrow x = e$$

$$f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$$

$$= \frac{-x - 2x(1 - \log x)}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

$$f''(e) == \frac{-3 + 2\log e}{e^3} = \frac{-3 + 2}{e^3} = \frac{-1}{e^3} < 0$$

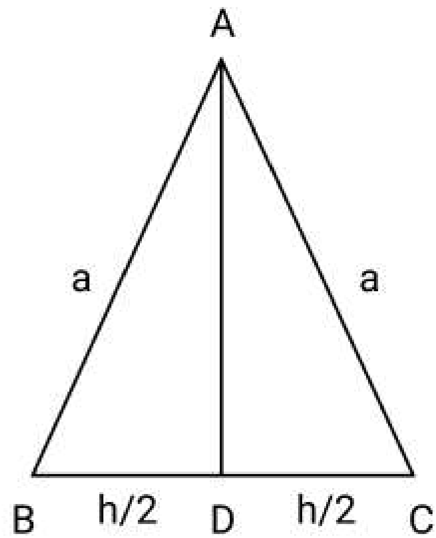
f is the maximum at $x = e$

2.

The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Ans - Let $\triangle ABC$ be isosceles where BCis the base of fixed length b. Also let the length of the two equal sides of $\triangle ABC$ be a.

Draw $AD \perp BC$.



$$AD = \sqrt{a^2 - \frac{b^2}{4}}$$

$$\text{Area of triangle } = \frac{1}{2} b \sqrt{a^2 - \frac{b^2}{4}}$$

$$\text{Area of triangle } = \frac{1}{2} b \sqrt{a^2 - \frac{b^2}{4}}$$

$$\frac{dA}{dt} = \frac{1}{2} b \cdot \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}} \frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}} \frac{da}{dt}$$

$$\frac{da}{dt} = -3 \text{ cm/s}$$

$$\therefore \frac{da}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

when $a = b$

$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3b}$$

3.

Find the intervals in which the function f given by

$$\mathbf{f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}}$$

is (i) increasing (ii) decreasing

Ans - $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$

$$\therefore f'(x) = \frac{(2 + \cos x)(4\cos x - 2 - \cos x + x\sin x) - (4\sin x - 2x - x\cos x)(-\sin x)}{(2 + \cos x)^2}$$

$$= \frac{(2 + \cos x)(3\cos x - 2 + x\sin x) + \sin x(4\sin x - 2x - x\cos x)}{(2 + \cos x)^2}$$

$$= \frac{6\cos x - 4 + 2x\sin x + 3\cos^2 x - 2\cos x + x\sin x\cos x + 4\sin^2 x - 2\sin^2 x - 2x\sin x - x\sin x\cos x}{(2 + \cos x)^2}$$

$$= \frac{4\cos x - 4 + 3\cos^2 x + 4\sin^2 x}{(2 + \cos x)^2}$$

$$= \frac{4\cos x - \cos^2 x}{(2 + \cos x)^2} = \frac{\cos x(4 - \cos x)}{(2 + \cos x)^2}$$

$$f'(x) = 0$$

$$\Rightarrow \cos x = 0 \cos x = 4$$

$$\cos x \neq 4$$

$$\cos x = 0$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

In $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right), f'(x) > 0$

f(x) is increasing for $0 < x < \frac{x}{2}$ and $\frac{3\pi}{2} < x < 2\pi$.

In $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right), f'(x) < 0$

f(x) is decreasing for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

4.

Find the intervals in which the function f given by

$$\mathbf{f(x) = x^3 + \frac{1}{x^3}, x \neq 0}$$

is (i) increasing (ii) decreasing

Ans - $f(x) = x^3 + \frac{1}{x^3}$

$$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$

$$f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = x = \pm 1$$

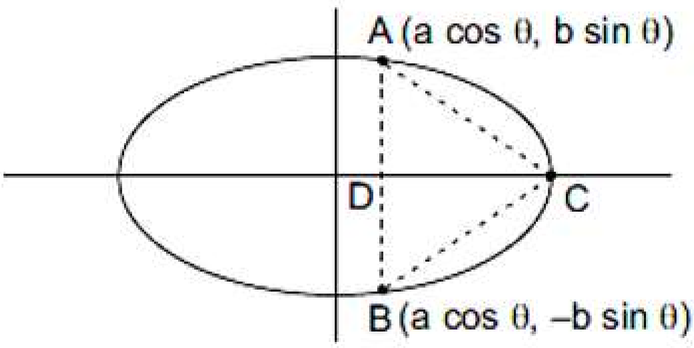
In $(-\infty, 1)$ and $(1, \infty)$ i.e..., when $x < -1$ and $x > 1, f'(x) > 0$. Thus, when $x < -1$ and $x > 1$, f is increasing.

In $(-1,1)$ i.e., when $-1 < x < 1, f'(x) < 0$. Thus, when $-1 < x < 1$, f is decreasing.

5.

Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

Ans –



Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let ABC, be triangle inscribed in ellipse where vertex C is at (a, 0). Since ellipse is symmetrical with x-axis and y-axis $y_1=\pm\frac{b}{a}\sqrt{a^2-x_1^2}$.

Coordinates of A are $\left(-x_1,\frac{b}{a}\sqrt{a^2-x^2 1}\right)$ and coordinates of B are $\left(x_1,-\frac{b}{a}\sqrt{a^2-x_1^2.}\right)$ Since point $(-x_1,y_1)$ lies on the ellipse, area of triangle ABC is

$$A = \frac{1}{2}\left| a\left(\frac{2b}{a}\sqrt{a^2 - x_1^2}\right) + (-x_1)\left(-\frac{b}{a}\sqrt{a^2 - x_1^2}\right) + (-x_1)\left(-\frac{b}{a}\sqrt{a^2 - x^2 1}\right)\right.$$

$$\Rightarrow A = ba\sqrt{a^2 - x^2 1} + x_1 \frac{b}{a}\sqrt{a^2 - x_1^2}$$

$$\therefore \frac{dA}{dx_1} = \frac{-2xb}{2\sqrt{a^2 - x_1^2}} + \frac{b}{a}\sqrt{a^2 - x_1^2} - \frac{2bx_1^2}{a^2\sqrt{a^2 - x_1^2}}$$

$$= \frac{b}{2\sqrt{a^2 - x_1^2}}\left[-x_1 a + (a^2 - x_1^2) - x_1^2\right]$$

$$\Rightarrow \frac{dA}{dx_1} = \frac{b(-2x_1^2 - x_1^2 + a^2)}{a\sqrt{a^2 - x_1^2}}$$

$$\frac{dA}{dx_1} = 0$$

$$\Rightarrow -2x_1^2 - x_1 a + a^2 = 0$$

$$\Rightarrow x_1 = \frac{a \pm \sqrt{a^2 - 4(-2)(a^2)}}{2(-2)}$$

$$= \frac{a \pm \sqrt{9a^2}}{-4} = \frac{a \pm 3a}{-4}$$

$$\Rightarrow x_1 = -a, \frac{a}{2}$$

x_1 cannot be equal to a.

$$\therefore x_1 = \frac{a}{2}$$

$$\Rightarrow y_1 = \frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}} = \frac{ba}{2a}\sqrt{3} = \frac{\sqrt{3}b}{2}$$

Now, $\frac{d^2A}{dx^21} = \frac{b}{a}\left\{\frac{\sqrt{a^2-x_1^2}(-4x_1-a)-(-2x_1^2-x_1a+a^2)\frac{(-2x_1)}{2\sqrt{a^2-x_1^2}}}{a^2-x_1^2}\right\}$

$$= \frac{b}{a}\left\{\frac{(a^2 - x_1^2)(-4x_1 - a) + x_1(-2x_1^2 - x_1 a + a^2)}{(a^2 - x_1^2)^{\frac{2}{3}}}\right\}$$

$$= \frac{b}{a}\left\{\frac{2x^3 - 3a^2x - a^3}{(a^2 - x_1^2)^{\frac{3}{2}}}\right\}$$

when $x_1 = \frac{a}{2}$,

$$\frac{d^2A}{dx_1^2} = \frac{b}{a}\left\{\frac{2\frac{a^3}{8} - 3\frac{a^3}{2} - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\} = \frac{b}{a}\left\{\frac{\frac{a^3}{4} - \frac{3}{2}a^3 - a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$= \frac{b}{a}\left\{\frac{\frac{9}{4}a^3}{\left(\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}\right)}\right\} < 0$$

Area is maximum when $x_1 = \frac{a}{2}$. Maximum area of the triangle is

$$A = b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}}$$

$$= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2}$$

$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

6.

A tank with a rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2m and volume is $8m^3$. If the building of the tank costs Rs 70 per sq meter for the base and Rs 45 per sq meter for the sides. What is the cost of the least expensive tank?

Ans - Let l, b and h represent length, breadth, and height of the tank respectively.

Given, height (h) $= 2\text{m}$ and volume of tank $= 8\ \text{m}^3$

Volume of the tank $= l \times b \times h = l \times b \times 2 = 8$

$$\Rightarrow lb = 4 \Rightarrow b = \frac{4}{l}$$

Area of the base $= lb = 4$

Area of 4 walls (A) $= 2h(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$

$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$

Now, $\frac{dA}{dl} = 0$

$$\Rightarrow l - \frac{4}{l^2} = 0$$

$$\Rightarrow l^2 = 4$$

$$\Rightarrow l = \pm 2$$

$$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$$

$$\frac{d^2A}{dl^2} = \frac{32}{l^3}$$

$l = 2, \frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0$

Area is the minimum when $l = 2$.

We have $l = b = h = 2$.

Cost of building base $= \text{Rs } 70x(lb) = \text{Rs } 70(4) = \text{Rs } 280$

Cost of building walls $= \text{Rs}2h(l + h) \times 45 = \text{Rs}90(2)(2 + 2) = \text{Rs } 8(90) = \text{Rs } 720$

Required total cost $= \text{Rs}(280 + 720) = \text{Rs } 1000$

7.

The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Ans - $2\pi r + 4a = k$ (where k is constant)

$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

Sum of areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k = 2\pi r)^2}{16}$$

$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(2\pi)}{16} = 2\pi r$$

$$= -\frac{\pi(k - 2\pi r)}{4}$$

Now, $\frac{dA}{dr} = 0$

$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$

$$8r = k - 2\pi r$$

$$\Rightarrow (8 + 2\pi)r = k$$

$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$

Now, $\frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$

$\therefore$ when $r = \frac{k}{2(4+\pi)}, \frac{d^2A}{dr^2} > 0.$

Area is least when $r = \frac{k}{2(4+\pi)}$ where $r = \frac{k}{2(4+\pi)}$,

$$a = \frac{k - 2\pi\left[\frac{k}{2(4 + \pi)}\right]}{4} = \frac{8k + 2\pi k - 2\pi k}{2(4 + \pi) \times 4} = \frac{k}{4 + \pi} = 2r$$

8.

A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Ans - x and y are length and breadth of rectangular window.

Radius of semi-circular opening $= \frac{x}{2}$

$$\therefore x + 2y + \frac{\pi x}{2} = 10$$

$$\Rightarrow x\left(1 + \frac{\pi}{2}\right) + 2y = 10$$

$$\Rightarrow 2y = 10 - x\left(1 + \frac{\pi}{2}\right)$$

$$\Rightarrow y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$

$$= x\left[5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right] + \frac{\pi}{8} \times x^2 = 5x - x^2\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{8} \times x^2$$

$$\therefore \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2} + \frac{\pi}{4}\right) + \frac{\pi}{4}x$$

$$\frac{d^2A}{dx^2} = -\left(1 - \frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

$$\frac{dA}{dx} = 0$$

$$\Rightarrow 5 - x\left(1 + \frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$

$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$

$$\Rightarrow x\left(1 + \frac{\pi}{4}\right) = 5$$

$$\Rightarrow x = \frac{5}{\left(1 + \frac{\pi}{4}\right)} = \frac{20}{\pi + 4}$$

$$x = \frac{20}{\pi + 4}, \frac{d^2A}{dx^2} < 0$$

Area is maximum when length $x = \frac{20}{\pi+4}$ m.

$$\Rightarrow y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right) = 5 - \frac{5(2 + \pi)}{\pi + 4} = \frac{10}{\pi + 4}\,\text{m}$$

Required dimensions - length $= \frac{20}{\pi+4}$ m and breadth $= \frac{10}{\pi+4}$ m.

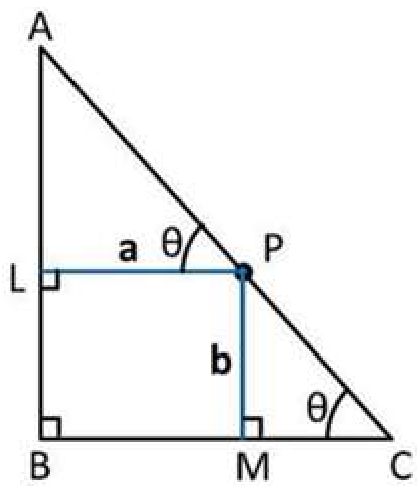
9.

A point of the hypotenuse of a triangle is at distance a and b from the sides of the triangle.
Show that the minimum length of the hypotenuse is
$\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

Ans – △ ABC right-angled at B.

AB = x and BC = y.

Let P be a point on hypotenuse such that P is at a distance of a and b from the sides AB and BC respectively.



$\angle c = \theta$

$AC = \sqrt{x^2 + y^2}$

$PC = b\text{cosec}\theta$

$AP = a\sec\theta$

$AC = AP + PC$

$AC = b\text{cosec}\theta + a\sec\theta \quad \ldots\ldots (1)$

$$\therefore \frac{d(AC)}{d\theta} = -b\text{cosec}\theta\cot\theta + a\sec\theta\tan\theta$$

$$\therefore \frac{d(AC)}{d\theta} = 0$$

$\Rightarrow a\sec\theta\tan\theta = b\text{cosec}\theta\cot\theta$

$$\Rightarrow \frac{a}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} = \frac{b}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$\Rightarrow a\sin^3\theta = b\cos^3\theta$

$\Rightarrow (a)^{\frac{1}{3}}\sin\ \theta = (b)^{\frac{1}{3}}\cos\ \theta$

$\Rightarrow \tan\ \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$

$\therefore \sin\theta = \frac{(b)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}$ and $\cos\ \theta = \frac{(a)^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}$

Clearly $\frac{d^2(AC)}{d\theta^2} < 0$ when $\tan\ \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

$\Rightarrow$ length of hypotenuse is maximum when $\tan\ \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Now, when $\tan\ \theta = \left(\frac{b}{a}\right)^{\frac{1}{3}}$

$$AC = \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}}$$

$$= a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}\left(b^{\frac{2}{3}} + a^{\frac{2}{3}}\right) = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$\therefore$ Maximum length of hypotenuse $= \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

10.

Find the points at which the function f given
$f(x) = (x-2)^4(x+1)^3$ has
(i) local maxima **(ii) local minima**
(iii) point of inflexion

Ans - $f(x) = (x-2)^4(x+1)^3$

$\therefore f'(x) = 4(x-2)^3(x+1)^3 + 3(x+1)^2(x-2)^4$

$= (x-2)^3(x+1)^2[4(x+1) + 3(x-2)]$

$= (x-2)^3(x+1)^2(7x-2)$

$f'(x) = 0 \Rightarrow x = -1$ and $x = \frac{2}{7}$ or $x = 2$

For x close to $\frac{2}{7}$ and to left of $\frac{2}{7}$, $f'(x) > 0$

For x close to $\frac{2}{7}$ and to right of $\frac{2}{7}$, $f'(x) > 0$.

$x = \frac{2}{7}$ is point of local minima.

As the value of x varies $f'(x)$ does not changes its sign.

$x = -1$ is point of inflexion.

11.

Find the absolute maximum and minimum values of the function f given by

$$\mathbf{f(x) = \cos^2 x + \sin x, x \in [0, \pi]}$$

Ans - $f(x) = \cos^2 x + \sin x$

$f'(x) = 2\cos x(-\sin x) + \cos x = -2\sin x\cos x + \cos x$

$f'(x) = 0$

$\Rightarrow 2\sin x\cos x = \cos x \Rightarrow \cos x(2\sin x - 1) = 0$

$\Rightarrow \sin x = \frac{1}{2}$ or $\cos x = 0$

$\Rightarrow x = \frac{\pi}{6}$, or $\frac{\pi}{2}$ as $x \in [0, \pi]$

$$f\left(\frac{\pi}{6}\right) = \cos^2 \frac{\pi}{6} + \sin \frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$$

$f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1$

$f(\pi) = \cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1$

$$f\left(\frac{\pi}{2}\right) = \cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2} = 0 + 1 = 1$$

Absolute maximum value of f is $\frac{5}{4}$ at $x = \frac{\pi}{6}$

Absolute minimum value of f is 1 at $x = 0, x = \frac{\pi}{2}$, and π.

12.

Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Ans - $V = \frac{1}{3}\pi R^2 h$

$BC = \sqrt{r^2 - R^2}$

$h = r + \sqrt{r^2 - R^2}$

$$\therefore V = \frac{1}{3}\pi R^2\left(r + \sqrt{r^2 - R^2}\right)$$

$$\Rightarrow V = \frac{1}{3}\pi R^2 r + \frac{1}{3}\pi R^2\sqrt{r^2 - R^2}$$

$$\frac{dV}{dR} = \frac{2}{3}\pi Rr + \frac{2\pi}{3}\pi R\sqrt{r^2 - R^2} + \frac{R^2}{3}\cdot\frac{(-2R)}{2\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi}{3}\pi R\sqrt{r^2 - R^2} - \frac{R^3}{3\sqrt{r^2 - R^2}}$$

$$= \frac{2}{3}\pi Rr + \frac{2\pi Rr(r^2 - R^2) - \pi R^3}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow \frac{dV}{dR} = \frac{2}{3}\pi Rr + \frac{2\pi Rr^2 - 3\pi Rr^3}{3\sqrt{r^2 - R^2}}$$

$$\frac{dV}{dR} = 0$$

$$\Rightarrow \frac{2\pi rR}{3} = \frac{3\pi R^2 - 2\pi Rr^2}{3\sqrt{r^2 - R^2}}$$

$$\Rightarrow 2r\sqrt{r^2 - R^2} = 3R^2 - 2r^2$$

$$\Rightarrow 4r^2(r^2 - R^2) = (3R^2 - 2r^2)^2$$

$$\Rightarrow 14r^4 - 4r^2R^2 = 9R^4 + 4r^4 - 12R^2r^2$$

$$\Rightarrow 9R^4 - 8r^2R^2 = 0$$

$$\Rightarrow 9R^2 = 8r^2$$

$$\Rightarrow R^2 = \frac{8r^2}{9}$$

$$\frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}(2\pi r^2 - 9\pi R^2) - (2\pi R^3 - 3\pi R^3)(-6R)\frac{1}{2\sqrt{r^2 - R^2}}}{9(r^2 - R^2)}$$

$$\Rightarrow \frac{d^2V}{dR^2} = \frac{2\pi r}{3} + \frac{3\sqrt{r^2 - R^2}(2\pi r^2 - 9\pi R^2) - (2\pi R^3 - 3\pi R^3)(3R)\frac{1}{2\sqrt{r^2 - R^2}}}{9(r^2 - R^2)}$$

when $R^2 = \frac{8r^2}{9}, \frac{d^2V}{dR^2} < 0.$

volume is the maximum when $R^2 = \frac{8r^2}{9}$. $R^2 = \frac{8r^2}{9}$,

$$h = r + \sqrt{r^2 - \frac{8R^2}{9}} = r + \sqrt{\frac{r^2}{9}} = r + \frac{r}{3} = \frac{4r}{3}$$

$\therefore$ height of the cone $= \frac{4r}{3}$

13.

Let f be a function defined on $[a, b]$ such that $f'(x) > 0$ for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).

Ans - Consider I be the interval (a, b)

Given that $f'(x) > 0$ for all x in an interval I. Consider $x_1, x_1, \in I$ with $x_1 < x_2$

By Lagrange's Mean Value Theorem, we have,

$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$ where $x_1 < c < x_2$

$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$ where $x_1 < c < x_2$

Now $x_1 < x_2 \Rightarrow x_2 - x_1 > 0 \quad \dots\dots (1)$

Also, $f'(x) > 0$ for all x in an interval I

$\Rightarrow f'(c) > 0$

From equation (1), $f(x_2) - f(x_1) > 0$

$\Rightarrow f(x_1) < f(x_2)$

Thus, for every pair of points $x_1, x_1, \in I$ with $x_1 < x_2$

$\Rightarrow f(x_1) < f(x_2)$

Therefore, f(x) is strictly increasing in I .

14.

Let f be a function defined on [a, b] such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).

Ans - $h = 2\sqrt{R^2 - r^2}$

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

$$\frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r(R^2 - r^2) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$\Rightarrow \frac{dV}{dr} = \frac{4\pi r R^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow 4\pi r R^2 - 6\pi r^3 = 0$

$\Rightarrow r^2 = \frac{2R^2}{3}$

$$\frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}(4\pi R^2 - 18\pi r^2) - (4\pi r R^2 - 6\pi r^3)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{(R^2 - r^2)}$$

$$= \frac{(R^2 - r^2)(4\pi R^2 - 18\pi r^2) + r(4\pi r R^2 - 6\pi r^3)}{3}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{3}$$

$$\Rightarrow \frac{d^2V}{dr^2} = \frac{3}{\left(R^2 - r^2\right)\frac{3}{2}}$$

$r^2 = \frac{2R^2}{3}, \frac{d^2V}{dr^2} < 0$

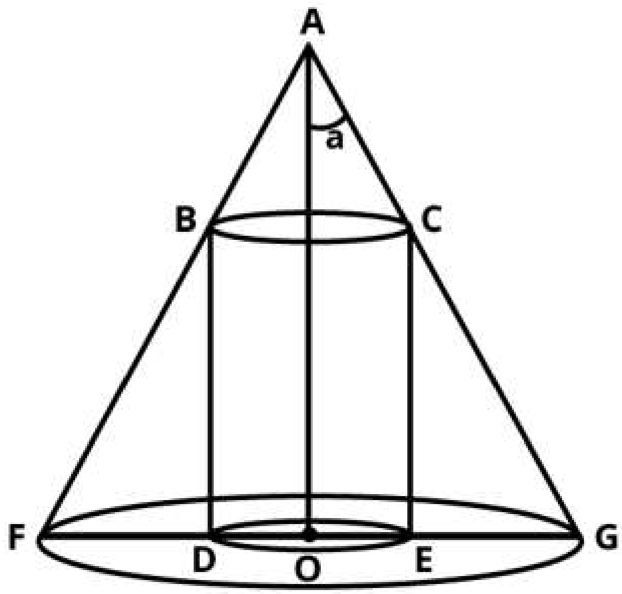
Volume is maximum when $r^2 = \frac{2R^2}{3}$

$\therefore$ Height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$.

15.

Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 a$

Ans -



$r = h \tan a$

since $\triangle$ AOG is similar to $\triangle$ CEG,

$$\frac{AO}{OG} = \frac{CE}{EG}$$

$$\Rightarrow \frac{h}{r} = \frac{H}{r - R}$$

$$\Rightarrow H = \frac{h}{r}(r - R)$$

$$\Rightarrow H = \frac{h}{h\tan a}(h\tan a - R)$$

$$\Rightarrow H = \frac{1}{\tan a}(h\tan a - R)$$

Volume of cylinder, $V = \pi R^2 H = \frac{\pi R^2}{\tan\ a}(h\tan\ a - R)$

$$\Rightarrow V = \pi R^2 h - \frac{\pi R^3}{\tan\ a}$$

$$\frac{dV}{dR} = 2\pi Rh \frac{3\pi R^2}{\tan\ a}$$

$$\frac{dV}{dR} = 0 \Rightarrow 2\pi Rh = \frac{3\pi R^2}{\tan\ a}$$

$$\Rightarrow 2h\tan\ a = 3R$$

$$\Rightarrow R = \frac{2h}{3}\tan\ a$$

$$\frac{d^2V}{dR^2} = 2\pi Rh - \frac{6\pi R}{\tan\ a}$$

When $R = \frac{2h}{3}\tan a$ we have,

$$\frac{d^2V}{dR^2} = 2\pi h - \frac{6\pi}{\tan\ a}\left(\frac{2h}{3}\tan\ a\right) = 2\pi h - 4\pi h = -2\pi h < 0$$

Volume of cylinder is greatest when $R = \frac{2h}{3}\tan\ a$.

$$\Rightarrow H = \frac{1}{\tan\ a}\left(h\tan\ a - \frac{2h}{3}\tan\ a\right) = \frac{1}{\tan\ a}\left(\frac{h\tan\ a}{3}\right) = \frac{h}{3}$$

Maximum volume of cylinder can be obtained as

$$V = \pi\left(\frac{2h}{3}\tan a\right)^2\left(\frac{h}{3}\right) = \pi\left(\frac{4h^2}{9}\tan^2 a\right)\left(\frac{h}{3}\right)$$

$$\Rightarrow V = \frac{4}{27}\pi h^3 \tan^2\ a$$

16.

A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic meters per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h **(B) 0.1 m/h**

(C) 1.1 m/h **(D) 0.5 m/h**

Ans - $V = \pi(\text{ radius })^2 \times \text{height}$

Given radius $r = 10$ m

$\Rightarrow V = \pi(10)^2 h = 100\pi h$

$$\frac{dV}{dt} = 100\pi \frac{dh}{dt}$$

Tank is being filled with wheat at rate of 314 cubic meters per hour.

$$\Rightarrow \frac{dV}{dt} = 314 \text{ m}^3/\text{h}$$

$$314 = 100\pi \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{314}{100(3.14)} = \frac{314}{314} = 1$$

The depth of wheat is increasing at 1 m/h.

The correct answer is **A**.