NCERT Solutions for Class 12 Maths Chapter 5 – Continuity and Differentiability

Exercise 5.1

1.

Prove that $f(x) = 5x - 3$ is a continuous function at $x=0$, $x=-3$ and $x=5$.

Ans - The given function is $f(x) = 5x - 3$

At $x = 0$,

$f(0) = 5 \times 0 - 3 = -3$

Taking limit as $x \to 0$ both sides of the function give

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (5x - 3) = 5 \times 0 - 3 = -3$

$\Rightarrow \lim_{x \to 0} f(x) = f(0)$

$\Rightarrow$ f satisfies continuity at $\mathbf{x = 0}$

At $x = -3$,

$f(-3) = 5 \times (-3) - 3 = -18$

Now, taking limit as $x \to 3$ both sides of the function give

$\lim_{x \to 3} f(x) = \lim_{x \to 3} (5x - 3) = 5 \times (-3) - 3 = -18$

$\Rightarrow \lim_{x \to 3} f(x) = f(-3)$

$\Rightarrow$ f satisfies continuity at $\mathbf{x = -3}$

At $x = 5$,

$f(x) = f(5) = 5 \times 5 - 3 = 25 - 3 = 22$

Taking limit as $x \to 5$ both sides of the function give

$\lim_{x\to5} f(x) = \lim_{x\to5} (5x - 3) = 5 \times 5 - 3 = 22$

$\Rightarrow \lim_{x\to5} f(x) = f(5)$

$\Rightarrow$ f satisfies continuity at $\mathbf{x = 5}$

2.

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

Ans - The given function is $f(x) = 2x^2 - 1$

At $x = 3$,

$f(3) = 2 \times 3^2 - 1 = 17$

Taking limit as $x \to 3$ both sides of the function give

$\lim_{x\to3} f(x) = \lim_{x\to3} (2x^2 - 1) = 2 \times 3^2 - 1 = 17$

$\Rightarrow \lim_{x\to5} f(x) = f(5)$

$\Rightarrow$ f satisfies continuity at $x = 3$.

3.

Examine the following functions for continuity.

(a) $\mathbf{f(x) = x - 5}$

(b) $\mathbf{f(x) = \frac{1}{x-5}}$

(c) $\mathbf{f(x) = \lim_{x \to c} \frac{x^2-25}{x+5}, x \neq 5}$

(d) $\mathbf{f(x) = |x - 5| = \begin{cases} 5 - x, \text{if } x < 5 \\ x - 5, \text{if } x \geq 5 \end{cases}}$

Ans - (a) The given function is $f(x) = x - 5$

It is assured that for every real number k, f is defined and its value at k is $k - 5$. Also, it can be noted that

$$\lim_{x \to k} f(x) = \lim_{x \to k} f(x - 5) = k = k - 5 = f(k)$$

$$\Rightarrow \lim_{x \to k} f(x) = f(k)$$

∴ f satisfies continuity at every real number and so, it is a continuous function.

(b) The given function is $f(x) = \frac{1}{x-5}$

Let $k \neq 5$ be any real number. Taking limit as $x \to k$ on both sides of the function give,

$$\lim_{x \to k} f(x) = \lim_{x \to k} f\left(\frac{1}{x-5}\right) = \frac{1}{k-5}$$

$f(k) = \frac{1}{k-5}$, since $k \neq 5$

$$\Rightarrow \lim_{x \to k} f(x) = f(k)$$

∴ f satisfies continuity at every point in the domain of f and so, it is a continuous function.

(c) The given function is $f(x) = \lim\limits_{x \to c} \frac{x^2-25}{x+5}, x \neq 5$

Let $c \neq 5$ be any real number. Taking limit as $x \to c$ on both sides of the function give,

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{x^2 - 25}{x + 5} = \lim_{x \to c} \frac{(x+5)(x-5)}{x+5}$$

$$= \lim_{x \to c}(x - 5) = (c - 5)$$

$f(c) = \frac{(c+5)(c-5)}{c+5} = (c - 5)$, since $c \neq 5$.

$\Rightarrow \lim\limits_{x \to c} f(x) = f(c)$

$\therefore$ f satisfies continuity at every point in the domain of f and so it is a continuous function.

(d) The given function is $f(x) = |x - 5| = \begin{cases} 5 - x, \text{if } x < 5 \\ x - 5, \text{if } x \geq 5 \end{cases}$

f is defined at all points in the real line. Let c be a point on a real line.

$\Rightarrow$ We have 3 cases:- $c < 5$ or $c = 5$ or $c > 5$.

Case(i): $c < 5$

$f(c) = 5 - c$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (5 - x) = 5 - c$

$\Rightarrow \lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} f(c)$

$\therefore$ f is continuous at all real numbers which are less than 5.

Case (ii): $c = 5$

$f(c) = f(5) = (5 - 5) = 0$

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5} (5 - x) = (5 - 5) = 0$

Similarly

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5} (x - 5) = 0$

$\Rightarrow \lim_{x \to c^-} f(x) = f(c).$

$\therefore$ f satisfies continuity at $x = 5$, and so f is continuous at $x = 5$.

4.

Prove that $f(x) = x^n$ is continuous at x=n, where n is a positive integer.

Ans - The given function is $f(x) = x^n$

Function f is defined at all positive integers n and its value at $x = n$ is n^n.

$\lim_{x \to n} f(n) = \lim_{x \to n} f(x^n) = n^n$

$\Rightarrow \lim_{x \to n} f(x) = f(n).$

$\Rightarrow$ Function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

5.

Is the function f defined by

$$f(x) = \begin{cases} x, \text{ if } x \le 1 \\ 5, \text{ if } x > 1 \end{cases}$$

continuous at $x = 0$, $x = 1$? At $x = 2$?

Ans - The given function is $f(x) = \begin{cases} x, \text{if } x \leq 1 \\ 5, \text{if } x > 1 \end{cases}$

Function f is defined at $x = 0$ and its value at $x = 0$ is 0.

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x = 0$

$\Rightarrow \lim\limits_{x \to 0} f(x) = f(0)$

∴ Function f is continuous at $x = 0$.

Function f is defined at $x = 1$ and its value at $x = 1$ is 1.

Left-hand limit of the function f at $x = 1$ is

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$

Similarly right-hand limit of the function f at $x = 1$ is

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} f(5)$

$\Rightarrow \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x).$

∴ Function f is not continuous at $x = 1$.

Function f is defined at $x = 2$ and its value at $x = 2$ is 5.

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} f(5) = 5.$

$\Rightarrow \lim\limits_{x \to 2} f(x) = f(2)$

∴ Function f is continuous at $x = 2$.

6.

Find all points of discontinuity of f, where f is defined by,

$$\mathbf{f(x) = \begin{cases} 2x + 3, \text{if } x \leq 2 \\ 2x - 3, \text{if } x > 2 \end{cases}}$$

Ans - The given function is $f(x) = \begin{cases} 2x+3, \text{if } x \le 2 \\ 2x-3, \text{if } x > 2 \end{cases}$

Function f is defined at all the points in the real line. Let us consider c be a point on the real line.

$\Rightarrow$ 3 cases will be there:- $c < 2$ or $c > 2$ or $c = 2$

Case (i), When $c < 2$

$f(c) = 2c + 3$

$\lim_{x \to c} f(x) = \lim_{x \to \infty} (2x + 3) = 2c + 3.$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\therefore$ Function f is continuous at all points x, where $x < 2$.

Case (ii), When $c > 2$

$f(c) = 2c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to \infty} (2x - 3) = 2c - 3$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\therefore$ Function f is continuous at all points x, where $x > 2$.

Case (iii), When $c = 2$

Left-hand limit of the function f at $x = 2$ is

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 2 \times 2 + 3 = 7$

Right-hand limit of the function f at $x = 2$ is,

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x + 3) = 2 \times 2 - 3 = 1.$

$\Rightarrow$ At $x = 2$, $\lim_{x \to 2^-} f(x) \ne \lim_{x \to 2^+} f(x)$.

$\therefore$ Function f is not continuous at $x = 2$

Hence, $x = 2$ is the only point of discontinuity of the function f(x).

7.

Find all points of discontinuity of f, where f is defined by,

$$\mathbf{f(x)=\begin{cases} |x|+3, \text{ if } x\leq -3 \\ -2x, \text{ if } -3<x<3 \\ 6x+2, \text{ if } x\geq 3 \end{cases}}$$

Ans - The given function is $f(x)=\begin{cases} |x|+3, \text{if } x\leq -3 \\ -2x, \text{if } -3<x<3 \\ 6x+2, \text{if } x\geq 3 \end{cases}$

Function f is defined at all the points in the real line. Let us assume c as a point on the real line.

$\Rightarrow$ 5 cases may arise:- $c<-3$ or $c=-3$ or $-3<c<3$, or $c=3$, or $c>3$.

Case (i), When $c<-3$

$f(c)=-c+3$

$\lim_{x\to c} f(x)=\lim_{x\to c}(-x+3)=-c+3$

$\Rightarrow \lim_{x\to c} f(x)=f(c)$

$\therefore$ Function f is continuous at all points x, where $x<-3$

Case (ii), When $c=-3$

$f(-3)=-(-3)+3=6$

Left-hand limit of the function f will be

$\lim_{x\to -3^-} f(x)=\lim_{x\to -3^-}(-x+3)=-(-3)+3=6$

Similarly right-hand limit of the function f will be

$\lim_{x\to -3^+} f(x)=\lim_{x\to -3^+}(-2x)=-2\times(-3)=6$

$\Rightarrow \lim_{x\to -3} f(x)=f(-3)$

$\therefore$ Function f is continuous at $x=-3$

Case (iii), When $-3 < c < 3$

$f(c) = -2c$

$\lim_{x \to c} f(x) = \lim_{x \to c} f(-2x) = -2c.$

$\Rightarrow \lim_{x \to c} f(x) = f(c).$

$\therefore$ Function f is continuous at x, where $-3 < x < 3$.

Case (iv), When $c = 3$

Left-hand limit of the function f at $x = 3$ is

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x) = -2 \times 3 = -6$ and

Similarly right-hand limit of the function at $x = 3$ is

$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (6x + 2) = 6 \times 3 + 2 = 20$

$\Rightarrow$ At $x = 3$, $\lim_{x \to 3^-} (x) \neq \lim_{x \to 3^+} f(x).$

$\therefore$ Function f is not continuous at $x = 3$

Case (v), When $c > 3$.

$f(c) = 6c + 2$

$\lim_{x \to c} f(x) = \lim_{x \to c} (6x + 2) = 6c + 2.$

$\Rightarrow \lim_{x \to c} f(x) = f(c).$

$\therefore$ Function f is continuous at all points x, when $x > 3$

Hence, $x = 3$ is the only point of discontinuity of the function f.

8.

Find all points of discontinuity of f, where f is defined by,

$$\mathbf{f(x) = \begin{cases} \frac{|x|}{x}, \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases}}$$

Ans - The given function is $f(x) = \begin{cases} \frac{|x|}{x}, \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases}$

f(x) can be rewritten as

$$f(x) = \begin{cases} \frac{|x|}{x} = \frac{-x}{x} = -1, \text{if } x < 0 \\ 0, \text{if } x = 0 \\ \frac{|x|}{x} = \frac{x}{x} = 1, \text{if } x > 0 \end{cases}$$

Function f is defined at all points of the real line. Let us assume c as a point on the real line

Three cases may anise, either $c < 0$, or $c = 0$, or $c > 0$.

Case (i), When $c < 0$.

$f(c) = -1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1.$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

∴ Function f is continuous at all the points x where $x < 0$.

Case (ii), When $c = 0$

Left-hand limit of the function f at $x = 0$ is

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$ and

Similarly right-hand limit of the function at $x = 0$ is

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1) = 1.$

At $x = 0$, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$.

∴ Function f is not continuous at $x = 0$.

Case (iii), When $c > 0$.

$f(c) = 1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(1) = 1.$

$\Rightarrow \lim_{x \to c} f(x) = f(c).$

$\therefore$ Function f is continuous at all the points x, for $x > 0$.

Hnece, $x = 0$ is the only point of discontinuity for the function f.

9.

Find all points of discontinuity of f, where f is defined by,

$$\mathbf{f(x) = \begin{cases} \frac{x}{|x|}, \text{if } x < 0 \\ -1, \text{if } x \geq 0 \end{cases}}$$

Ans - The given function is $f(x) = \begin{cases} \frac{x}{|x|}, \text{if } x < 0 \\ -1, \text{if } x \geq 0 \end{cases}$

We know that, if $x < 0$, then $|x| = -x$

$$\Rightarrow f(x) = \begin{cases} \frac{|x|}{x} = \frac{x}{-x} = -1 \text{ if } x < 0 \\ -1, \quad \text{if } x = 0 \\ -1, \quad \text{if } x > 0 \end{cases}$$

$\Rightarrow f(x) = -1$ for all positive real numbers.

Assume c as any real number.

$\Rightarrow$ 3 cases may arise, either $c < 0$, or $c = 0$, or $c > 0$.

Case (i), When $c < 0$

$f(c) = -1$ and

$\lim_{x \to c} f(x) = \lim_{x \to c} (-1) = -1.$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\therefore$ Function f is continuous at all the points x where $x < 0$.

Case (ii), When $c = 0$

Left-hand limit of the function f at $x = 0$ is

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$ and

Similarly right-hand limit of the function at $x = 0$ is

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (-1) = -1.$

At $x = 0$, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

$\therefore$ Function f is continuous at $x = 0$

Case (iii), When $c > 0$.

$f(c) = -1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(-1) = -1$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

∴ Function f is continuous at all the points x, for $x > 0$

∴ the function f(x) is a continuous function. Hence, there does not exist any point of discontinuity.

10.

Find all points of discontinuity of f, where f is defined by,

$$\mathbf{f(x) = \begin{cases} x + 1, \text{if } x \geq 1 \\ x^2 + 1, \text{if } x < 1 \end{cases}}$$

Ans - The given function is $f(x) = \begin{cases} x + 1, \text{if } x \geq 1 \\ x^2 + 1, \text{if } x < 1 \end{cases}$

Function f(x) is defined at all the points of the real line. Assume c as a point on the real line.

⇒ 3 cases may will be there, either $c < 1$, or $c = 1$, or $c > 1$.

Case (i), When $c < 1$.

$f(c) = c^2 + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c}(x^2 + 1) = c^2 + 1$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

∴ Function f is continuous at all the points x, where $x < 1$.

Case (ii), When c = 1

$f(c) = f(1) = 1 + 1 = 2$

Left-hand limit of f at x = 1 is

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2$

Similarly right-hand limit of the function at x = 1 is

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 1 + 1 = 2.$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

∴ Function f is continuous at x = 1.

Case (iii), When c > 1.

Then, we have $f(c) = c + l1$and

$\lim_{x \to c} f(x) = \lim_{x \to c} (x + 1) = c + 1.$

Therefore,

$\lim_{x \to c} f(x) = f(c).$

∴ Function f is continuous at all the points x, where x > 1.

Hence, there does not exist any discontinuity points.

11.

Find all points of discontinuity of f, where f is defined by,

$$\mathbf{f(x) = \begin{cases} x^3 - 3, & \text{if } x \le 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}}$$

Ans - The given function is $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$

Function f is defined at all points in the real line. Let us assume c as a point on the real line.

Case (i), When $c < 2$

$f(c) = c - 3$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 - 3) = c^3 - 3.$

∴ Function f is continuous at all the points x, where $x < 2$

Case (ii), When $c = 2$

$f(c) = f(2) = 2^3 - 3 = 5$

Left-hand limit of the function is

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3) = 2^3 - 3 = 5$

Similarly right-hand limit of the function is

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1) = 2^2 + 1 = 5.$

$\Rightarrow \lim_{x \to 2} f(x) = f(2).$

∴ Function f is continuous at $x = 2$

Case (iii), When $c > 2$

$f(c) = c^2 + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1.$

$\Rightarrow \lim_{x \to c} f(x) = f(c).$

∴ Function f is continuous at all the points x, where $x > 2$.

Thus, the function f is continuous at all the points on the real line. Hence, f does not have any point of discontinuity.

12.

Find all points of discontinuity of f, where f is defined by,

$$\mathbf{f(x)} = \begin{cases} \mathbf{x^{10} - 1,} & \mathbf{if\ x \leq 1} \\ \mathbf{x^2, if\ x > 12} \end{cases}$$

Ans - The given function is $f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, \text{if } x > 12 \end{cases}$

Function f is defined at every point of the real line. Let us assume c as a point on the real number line.

Case (i), When $c < 1$.

$f(c) = c^{10} - 1$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (x^{10} - 1) = c^{10} - 1$

$\Rightarrow \lim\limits_{x \to c} f(x) = f(c)$

$\therefore$ Function f is continuous at every point x, for $x < 1$.

Case (ii), When $c = 1$

Left-hand limit of the function $f(x)$ at $x = 1$ is

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^{10} - 1) = 1^{10} - 1 = 1 - 1$ and

Similarly right-hand limit of the function f at $x = 1$ is

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x^2) = 1^2 = 1.$

So, we can notice that, $\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$.

$\therefore$ Function f is not continuous at $x = 1$

Case (iii), When $c > 1$

$f(c) = c^2$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2) = c^2$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\therefore$ Function f is continuous at every point x, for $x > 1$.

Hence, we can conclude that $x = 1$ is the only point of discontinuity for the function f.

13.

Is the function defined by $\mathbf{f(x) = \begin{cases} x + 5, & \text{if } x \le 1 \\ x - 5, & \text{if } x > 1 \end{cases}}$ a continuous function?

Ans - The given function is $f(x) = \begin{cases} x + 5, & \text{if } x \le 1 \\ x - 5, & \text{if } x > 1 \end{cases}$

Function f is defined at every point on the real line. Let us assume c as a point on the real line.

Case (i), When $c < 1$

$f(c) = c + 5$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x + 5) = c + 5$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\therefore$ Function f is continuous at every point x, for $x < 1$

Case (ii), When $c = 1$.

$f(1) = 1 + 5 = 6$

Left-hand limit of the function f at $x = 1$ is

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 5) = 1 + 5 = 6$

Right-hand limit of the function at $x = 1$ is

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 5) = 1 - 5 = 4$

$\Rightarrow \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x) f(x).$

$\therefore$ Function f is not continuous at $x = 1$.

Case (iii), When $c > 1$.

$f(c) = c - 5$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x + 5) = c - 5$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\therefore$ Function f is continuous at every point x, for $x > 1$.

Hence, we can conclude that $x = 1$ is the only point of discontinuity for the function f.

14.

Discuss the continuity of the function f, where f is defined by

$$\mathbf{f(x) = \begin{cases} 3, \text{if } x \leq x \leq 1 \\ 4, \quad \text{if } 1 < x < 3 \\ 5, \text{if } 3 \leq x \leq 10 \end{cases}}$$

Ans - The given function is $f(x) = \begin{cases} 3, \text{if } x \leq x \leq 1 \\ 4, \text{ if } 1 < x < 3 \\ 5, \text{if } 3 \leq x \leq 10 \end{cases}$

Function f is defined in the interval [0, 10]. Let us assume c as a point in the interval [0, 10].

Then there may arise five cases

Case (i), When $0 < c < 1$.

$f(c) = 3$.

$$\lim_{x \to c} f(x) = \lim_{x \to c} (3) = 3$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

$\therefore$ Function f is continuous at the interval [0, 1].

Case (ii), When $c = 1$.

$f(3) = 3$

Left-hand-limit of the function at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3) = 3$$

Right-hand limit of the function at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4) = 4$$

$$\Rightarrow \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x).$$

$\therefore$ Function f is not continuous at $x = 1$.

Case (iii), When $1 < c < 3$

$f(c) = 4$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (4) = 4.$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

$\therefore$ Function f is continuous at every point in interval [1, 3].

Case (iv), When c = 3.

f(c) = 5

Left-hand-limit of the function f at x = 3 is

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} (4) = 4$$

Right-hand limit of the function at x = 3 is

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} (5) = 5.$$

$$\Rightarrow \lim_{x\to3^-} f(x) \neq \lim_{x\to3^+} f(x).$$

∴ Function f is not continuous at x = 3.

Case (v), When 3 < c < 10.

f(c) =5

$$\lim_{x\to c} f(x) = \lim_{x\to c} (5) = 5.$$

$$\Rightarrow \lim_{x\to c} f(x) = f(c)$$

∴ Function f is continuous at every point in interval [3, 10]

Hence, the function f is not continuous at x = 1 and x = 3.

15.

Discuss the continuity of the function f, where f is defined by

$$\mathbf{f(x)} = \begin{cases} \mathbf{2x, \text{ if } x < 0} \\ \mathbf{0, \text{ if } 0 \le x \le 1} \\ \mathbf{4x, \text{ if } x > 1} \end{cases}$$

Ans - The given function is $f(x) = \begin{cases} 2x, \text{if } x < 0 \\ 0, \text{if } 0 \leq x \leq 1 \\ 4x, \text{if } x > 1 \end{cases}$

Let us consider c be a point on the real number line. Then, five cases will be there.

Case (i), When $c < 0$

$f(c) = 2c$

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c.$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\therefore$ Function f is continuous at every point x whenever $x < 0$.

Case (ii), When $c = 0$

$f(c) = f(0) = 0$

Left-hand-limit of the function f at $x = 0$ is

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x) = 0$

Right-hand limit of the function at $x = 0$ is,

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (0) = 0.$

$\Rightarrow \lim_{x \to 0} f(x) = f(0).$

$\therefore$ Function f is continuous at $x = 0$

Case (iii), When $0 < c < 1$

$f(x) = 0$

$\lim_{x \to c} f(x) = \lim_{x \to c} (0) = 0$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\therefore$ Function f is continuous at every point in interval $(0, 1)$.

Case (iv), When $c = 1$

$f(c) = f(1) = 0$

Left-hand-limit of the funtion at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (0) = 0$$

Right-hand limit of the function at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x) = 4 \times 1 = 4.$$

$$\Rightarrow \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x).$$

$\therefore$ Function f is not continuous at $x = 1$.

Case (v), When $c > 1$

$f(c) = f(1) = 0$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (4x) = 4c$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

$\therefore$ Function f is continuous at every point x, for $c > 1$.

Hence, the function f is discontinuous only at $x = 1$.

16.

Discuss the continuity of the function f, where f is defined by

$$\mathbf{f(x) = \begin{cases} -2, \text{if } x \leq -1 \\ 2x, \text{if } -1 < x \leq 1 \\ 2, \quad \text{if } x > 1 \end{cases}}$$

Ans - The given function is $f(x) = \begin{cases} -2, \text{if } x \le -1 \\ 2x, \text{if} -1 < x \le 1 \\ 2, \text{ if } x > 1 \end{cases}$

Function f is defined at every point in the interval $[-1, \infty)$.
Let us assume c is a point on the real number line

Case (i), When $c < -1$

$f(c) = -2$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (-2) = -2.$

$\Rightarrow \lim\limits_{x \to c} f(x) = f(c)$

∴ Function f is continuous at every point x for x – 1.

Case (ii), When $c > 2$

$f(c) = 2c - 3$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to \infty} (2x - 3) = 2c - 3$

$\Rightarrow \lim\limits_{x \to c} f(x) = f(c)$

∴ Function f is continuous at all points x, where $x > 2$.

Case (iii), When $-1 < c < 1$

$f(c) = 2c$

$\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} (2x) = 2c.$

$\Rightarrow \lim\limits_{x \to c} f(x) = f(c)$

∴ Function f is continuous at every point in interval $(-1, 1)$.

Case (iv), When c = 1.

$f(c) = f(1) = 2 \times 1 = 2$

Left-hand-limit of the function at x = 1 is

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x) = 2 \times 1 = 2$

Right-hand limit of the function at x = 1 is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2.$

$\Rightarrow \lim_{x \to 1} f(x) = f(c).$

∴ Function f is continuous at x = 2.

Case (v), When c > 1.

$f(c) = 2$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (2) = 2.$

$\Rightarrow \lim_{x \to 1} f(x) = f(c).$

∴ Function f is continuous at every point x, for x > 1.

Hence, it can be concluded that the function f is continuous for all points.

17.

Find the relationship between a and b so that the function f defined by $f(x) = \begin{cases} ax + 1, \text{ if } x \leq 3 \\ bx + 3, \text{ if } x > 3 \end{cases}$ is continuous at x=3

Ans - The given function is $f(x) = \begin{cases} ax + 1, \text{if } x \le 3 \\ bx + 3, \text{if } x > 3 \end{cases}$

The function f will be continuous at $x = 3$ if

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} f(x) = f(3)$, ... (1)

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1) = 3a + 1,$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} f(bx + 1) = 3b + 3,$ ... . (2)

and

$f(3) = 3a + 1$ ... (3)

$\therefore$ from (1), (2), and (3) we get,

$3a + 1 = 3b + 3$

$\Rightarrow 3a = 3b + 2$

$\Rightarrow a = b + \frac{2}{3}$

Hence, the required relationship between a and b is given by $a = b + \frac{2}{3}$.

18.

For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2 - 2x), \text{if } x \le 0 \\ 4x + 1, \text{if } x > 0 \end{cases}$ is continuous at x=0. Also discuss the continuity of f at x=1?

Ans - The given function is $f(x) = \begin{cases} \lambda(x^2 - 2x), \text{if } x \le 0 \\ 4x + 1, \text{if } x > 0 \end{cases}$

Function f will be continuous at x = 0 if

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0).$

Also, the R.H.L and L.H.L are given by,

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} f(4x + 1) = 4(0) + 1 = 1,$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \lambda(x^2 - 2x) = \lambda(0^2 - 2 \times 0) = 0$

So $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x).$

Hence, there does not exist any value of λ for which f is continuous at x = 0.

Now, at x = 1,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$\lim_{x\to 1} f(4x + 1) = 4 \times 1 + 1 = 5$

$\Rightarrow \lim_{x\to 1} f(4x + 1) = f(1).$

Hence, function f is continuous at x = 1, for all values of λ.

19.

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point, here [x] denotes the greatest integer value of x that are less than or equal to x.

Ans - The given function is $g(x) = x - [x]$

Function is defined at every integral point. Let us assume that n is an integer.

Then, $g(n) = n - [n] = n - n = 0$

Left-hand-limit as $x \to n$ to the function g gives

$$\lim_{x \to n^-} g(x) = \lim_{x \to n^-} \left[x - [x]\right] = \lim_{x \to n^-} (x) - \lim_{x \to n^-} [x]$$
$$= n - (n - 1) = 1$$

Right-hand-limit on the function at $x = n$ is

$$\lim_{x \to n^+} g(x) = \lim_{x \to n^+} \left[x - [x]\right] = \lim_{x \to n^+} (x) - \lim_{x \to n^+} [x] = n - n = 0$$

$$\Rightarrow \lim_{x \to n^-} g(x) \neq \lim_{x \to n^+} g(x).$$

$\therefore$ Function f is not continuous at $x = n$,

Hence, the function g is not continuous at any integral point.

20.

Is the function defined by $f(x) = x^2 - \sin x + 5$ is continuous at $x = \pi$.

Ans - The given function is $f(x) = x^2 - \sin x + 5$

At $x = \pi$,

$f(x) \; = f(\pi) \; = \pi^2 - \sin x + 5 \; = \pi^2 - 0 + 5 = \pi^2 + 5$

Taking limit as $x \to \pi$ on the function f(x) gives

$\lim_{x\to\pi} f(x) = \lim_{x\to\pi} (\pi^2 - \sin x + 5)$.

Now substitute $x = \pi \; + h$ into the function f(x).

When $x \to \pi$, then $h \to 0$

$\Rightarrow \lim_{x\to\pi} f(x) = \lim_{x\to\pi} (\pi^2 - \sin x) + 5.$

$= \lim_{h\to 0} [(\pi + h^2) - \sin(\pi + h) + 5]$

$= \lim_{h\to 0} (\pi + h)^2 - \lim_{h\to 0} \sin(\pi + h) + \lim_{h\to 0} 5$

$= (\pi + 0)^2 - \lim_{h\to 0}[\sin\pi . \cos h + \cos\pi . \sin h] + 5$

$= \pi^2 - \lim_{h\to 0} (\sin\pi . \cos h) - \lim_{h\to 0}(\cos\pi . \sin h) + 5$

$= \pi^2 - \sin\pi . \cos 0 - \cos\pi . \sin 0 + 5.$

$= \pi^2 - 0 \times 1 - (-1) \times 0 + 5 = \pi^2 + 5$

$\Rightarrow \lim_{x\to x} f(x) = f(\pi).$

Hence, it is concluded that function f is continuous at $x = n$.

21.

Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$ **(b) $f(x) = \sin x - \cos x$**
(c) $f(x) = \sin x \times \cos x$

Ans - It is known that if two functions g and h are continuous, then g+h, g-hand, g, h are also continuous.

Let us assume that, $g(x) = \sin x$ and $h(x) = \cos x$ are two continuous functions.

As $g(x) = \sin x$ is defined for every real number, so let c be a real number. Substitute $x = c + h$ into the function g.

When $x \to c$, then $h \to 0$.

$g(c) = \sin c$

$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$

$= \lim_{h \to 0} \sin(c + h)$

$= \lim_{h \to 0} [\sin c . \cos h + \cos c . \sin h]$

$= \lim_{h \to 0} (\sin c . \cos h) + \lim_{h \to 0} (\cos c . \sin h)$

$= \sin c . \cos 0 + \cos c . \sin 0$

$= \sin c + 0 = \sin c$

$\Rightarrow \lim_{x \to c} g(x) = g(c).$

$\therefore$ Function g is continuous.

Now let us assume that $h(x) = \cos x$

Function $h(x) = \cos x$ is defined for every real number. Let c be a real number

Substitute $x = c + h$ into the function.

When $x \to c$, then $h \to 0$

$h(c) = \cos c$

$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$

$= \lim_{h \to 0} \cos(c + h)$

$= \lim_{h \to 0} [\cos c. \cos h - \sin c. \sin h]$

$= \lim_{h \to 0} (\cos c. \cos h) - \lim_{h \to 0} (\sin c. \sin h)$

$= \cos c. \cos 0 - \sin c. \sin 0$

$= \cos c \times 1 - \sin c \times 0$

$= \cos c$

$\Rightarrow \lim_{h \to 0} h(x) = h(c)$

$\therefore$ Function h is continuous

Hence, we conclude that all the following functions are continuous.

(a) $f(x) = g(x) + h(x) = \sin x + \cos x$

(b) $f(x) = g(x) - h(x) = \sin x - \cos x$

(c) $f(x) = g(x) \times h(x) = \sin x \times \cos x$

22.

Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

Ans - We know that if two functions say g and h are continuous, then

i. $\frac{h(x)}{g(x)}, g(x) \neq 0$ is continuous.

ii. $\frac{1}{g(x)}, g(x) \neq 0$ is continuous.

iii. $\frac{1}{h(x)}, h(x) \neq 0$ is continuous.

Function $g(x) = \sin x$ is defined for all real numbers. Let us consider c to be a real number and substitute $x = c + h$ into the function g.

When $x \to c$, then $h \to 0$.

$g(c) = \sin c$

$\lim_{x \to c} g(x) = \lim_{x \to c} \sin x$

$= \lim_{h \to 0} \sin(c + h)$

$= \lim_{h \to 0} [\sin c. \cos h + \cos c. \sin h]$

$= \lim_{h \to 0} (\sin c. \cos h) += \lim_{h \to 0} (\cos c. \sin h)$

$= \sin c. \cos 0 + \cos c. \sin 0$

$= \sin c + 0 = \sin c$

$\Rightarrow \lim_{x \to c} g(x) = g(c)$

$\therefore$ Function $g(x) = \sin x$ is continuous

Now let $h(x) = \cos x$

Function $h(x) = \cos x$ is defined for all real numbers. Let us consider c to be a real number and substitute $x = c + h$ into the function h

$h(c) = \cos c$

$\lim_{x \to c} h(x) = \lim_{x \to c} \cos x$

$= \lim_{h \to 0} \cos(c + h)$

$= \lim_{h \to 0} [\cos c. \cos h - \sin c. \sin h]$

$= \lim_{h \to 0} (\cos c. \cos h) - \lim_{h \to 0} (\sin c. \sin h)$

$= \cos c. \cos 0 - \sin c. \sin 0$

$= \cos c \times 1 - \sin x \times 0$

$= \cos c$

$\Rightarrow \lim_{x \to c} h(x) = h(c)$

$\therefore$ Function $h(x) = \cos x$ is continuous

$\operatorname{cosec} x = \frac{1}{\sin x}$, $\sin x \neq 0$ is a continuous function.

Hence, the cosecant function is continuous except at $x = n\pi$, $n \in Z$.

$\sec x = \frac{1}{\cos x}$, $\cos x \neq 0$ is a continuous function.

Hence, secant function is also continuous except at $x = (2n+1)\frac{\pi}{2}$, $n \in Z$.

$\cot x = \frac{\cos x}{\sin x}$, where $\sin x \neq 0$ is a continuous function.

Hence, the cotangent function is continuous except at $x = n\pi$, $n \in Z$.

23.

Find all points of discontinuity of f, where

$$\mathbf{f(x) = \begin{cases} \dfrac{\sin x}{x}, \text{if } x < 0 \\ x + 1, \text{if } x \geq 0 \end{cases}}$$

Ans - The given function is $f(x) = \begin{cases} \frac{\sin x}{x}, \text{if } x < 0 \\ x + 1, \text{if } x \geq 0 \end{cases}$

Function f is defined at every point on the real number line.
Let us consider c be a real number.

There will be 3 cases, either $c < 0$, or $c > 0$, or $c = 0$.

Case (i), When $c < 0$

$$f(c) = \frac{\sin c}{c}$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(x + 1) = c + 1$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

∴ Function f is continuous at every point x, for $x < 0$.

Case (ii), When $c > 0$

$$f(c) = c + 1$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x + 1) = c + 1$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

∴ Function f is continuous at every point, where $x > 0$.

Case (iii), When $c = 0$.

$f(c) = f(0) = 0 + 1 = 1$

Left-hand-limit of the function f at $x = 0$ is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$$

Right-hand-limit of the function f at $x = 0$ is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + 1) = 1$$

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

∴ Function f is continuous at $x = 0$

∴ Function f is continuous at every real point. Hence, the function f does not have any point of discontinuity.

24.

Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases}$$

is a continuous function?

Ans - The given function is $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases}$

Function f is defined at every point on the real number line.
Let us consider c to be a real number.

Then, there will be 2 cases, either $c \neq 0$ or $c = 0$.

Case (i), When $c \neq 0$

$f(c) = c^2 \sin\frac{1}{c}$

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^2 \sin\frac{1}{x}\right) = \left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c} \sin\frac{1}{x}\right) = c^2 \sin\frac{1}{c}.$$

$\Rightarrow \lim_{x\to c} f(x) = f(c)$

$\therefore$ Function f is continuous at every point $x \neq 0$.

Case (ii), When $c = 0$

$f(0) = 0$ and also

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right)$$

We know that, $-1 \leq \sin\frac{1}{x} \leq 1, x \neq 0$.

$$\Rightarrow -x^2 \leq \sin\frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x\to 0}(-x^2) \leq \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \leq 0$$

$$\Rightarrow 0 \leq \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = 0$$

Similarly we have,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0} \left(x^2 \sin\frac{1}{x}\right) = 0$$

Therefore, $\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$.

∴ Function f is continuous at the point $x = 0$

∴ Function f is continuous at all real points. Hence, the function f is continuous.

25.

Examine the continuity of f, where f is defined by

$$\mathbf{f(x) = \begin{cases} \sin x - \cos x \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}}$$

Ans - The given function is $f(x) = \begin{cases} \sin x - \cos x \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$

Function f is defined at every point on the real number line.
Let us consider c to be a real number

Then, there will be 2 cases, either c f 0 or c=0

Case (i), When $c \neq 0$

$f(c) = \sin c - \cos c$

$\lim_{x\to c} f(x) = \lim_{x\to c} (\sin x - \cos x) = \sin c - \cos c$

$\Rightarrow \lim_{x\to c} f(x) = f(c)$

∴ Function f is continuous at every point x for $x \neq 0$

Case (ii), When $c = 0$

$f(0) = -1$

Left-hand-limit of the function f at $x = 0$ is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (\sin x - \cos x)$$

$$= \sin 0 - \cos 0 = 0 - 1 = -1$$

Right-hand-limit of the function f at $x = 0$ is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (\sin x - \cos x)$$

$$= \sin 0 - \cos 0 = 0 - 1 = -1$$

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

∴ Function f is continuous at $x = 0$.

∴ Function f is continuous at all real points. Hence, the function f is continuous.

26.

Find the values of k so that the function f is continuous at the indicated point in $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, \text{ if } x \neq \frac{\pi}{2} \\ 3, \text{ if } x = \frac{\pi}{2} \end{cases}$

Ans - The given function is $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, \text{if } x \neq \frac{\pi}{2} \\ 3, \text{if } x = \frac{\pi}{2} \end{cases}$

Function f is defined and continuous at $x = \frac{\pi}{2}$, since the value of the f at $x = \frac{\pi}{2}$ is equal with the limiting value of f at $x = \frac{\pi}{2}$.

Since, f is defined at $x = \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = 3$, so

$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x}$.

Substitute $x = \frac{\pi}{2} + h$ into the function f (x).

So, we have, $x \to \frac{\pi}{2} \Rightarrow h \to 0$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = \lim_{h \to 0} \frac{k\cos x\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$\Rightarrow k \lim_{h \to 0} \frac{-\sin h}{-2h} = \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2}.1 = \frac{k}{2}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{k}{2} = 3 \Rightarrow k = 6$$

Hence, the value of k is 6 for which the function f is continuous.

27.

Find the values of k so that the function f is continuous at the indicated point in $f(x) = \begin{cases} kx^2, \text{if } x \leq 2 \\ 3, \text{if } x > 2 \end{cases}$

Ans - The given function is $f(x) = \begin{cases} kx^2, \text{if } x \le 2 \\ 3, \text{if } x > 2 \end{cases}$

Function f is continuous at x = 2 only if f is defined at x = 2 and if the value of f at x =2 is equal with the limiting value of f at x = 2.

Left-hand-limit and right-hand-limit of f(x) at x = 2 are,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (kx^2) = k(2)^2 = 4k$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} f(x) = (3) = 3.$$

Since, the function is continuous at x = 2, so

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$\Rightarrow 4k = 3$

$\Rightarrow k = \frac{3}{4}$

Hence, the value of k is $\frac{3}{4}$ for which the function f is continuous.

28.

Find the values of k so that the function f is continuous at the indicated point in $\mathbf{f(x) = \begin{cases} kx + 1, \text{if } x \le \pi \\ \cos x, \text{if } x > \pi \end{cases}}$

Ans - The given function is $f(x) = \begin{cases} kx + 1, \text{if } x \leq \pi \\ \cos x, \text{if } x > \pi \end{cases}$

Function f is continuous at $x = \pi$ only if value of f at $x = \pi$ is equal with the limiting value of f at $x = \pi$.

$f(\pi)=k\pi+1$

Left-hand-limit of f(x)

$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} (kx + 1) = k\pi + 1$$

Right-hand-limit of f(x)

$$\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} \cos x = \cos \pi = -1$$

Since, the function f is continuous, so

$$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x) = f(\pi)$$

$\Rightarrow k\pi + l = -1$

$\Rightarrow k\pi - 2$

$\Rightarrow k = -\frac{2}{\pi}$

Hence, the value of k is for which the function f is continuous at $x = \pi$

29.

Find the values of k so that the function f is continuous at the indicated point in $f(x) = \begin{cases} kx + 1, \text{if } x \leq 5 \\ 3x - 5, \text{if } x > 5 \end{cases}$

Ans - The given function is $f(x) = \begin{cases} kx + 1, \text{if } x \le 5 \\ 3x - 5, \text{if } x > 5 \end{cases}$

Function f is continuous at $x = 5$ only if value of f at $x = 5$ is equal to the limiting value of f at $x = 5$.

$f(5) = kx + 1 = 5k + 1$

Left-hand-limit of the function,

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (kx + 1) = 5k + 1$$

Right-hand-limit of the function,

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (3x - 5) = 3(5) - 5 = 15 - 5 = 10$$

Since, function f is continuous, so

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} (3x - 5) = 5k + 1$$

$\Rightarrow 5k + 1 = 10$

$\Rightarrow 5k = 9$

$\Rightarrow k = \frac{9}{5}$

Hence, the value of k is $\frac{9}{5}$ for which the function f is continuous at x=5

30.

Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, \text{if } x \le 2 \\ ax + b, \text{if } 2 < x < 10 \\ 21, \text{if } x \ge 10 \end{cases}$$

is a continuous function.

Ans - The given function is $f(x) = \begin{cases} 5, \text{if } x \leq 2 \\ ax + b, \text{if } 2 < x < 10 \\ 21, \text{if } x \geq 10 \end{cases}$

Function f is defined at every point on the real number line.
i.e. if the function f is continuous then f is continuous at every real number.

So let f be continuous at x = 2 and x = 10.

At x = 2,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-} (5) = \lim_{x\to 2^+} (ax + b) = 5$$

$$\Rightarrow 5 = 2a + b$$

$$\Rightarrow 2a + b = 5 \qquad \ldots (1)$$

At x = 10,

$$\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$$

$$\Rightarrow \lim_{x\to 10^-} (ax + b) = \lim_{x\to 10^+} (21) = 21$$

$$\Rightarrow 10a + b = 21 \qquad \ldots\ldots (2)$$

Subtracting the equation (1) from the equation (2), gives

$$8a = 16 = a = 2$$

Substituting a = 2 in the equation (1), gives

$$2 \times 2 + b = 5$$

$$\Rightarrow 4 + b = 5 \Rightarrow b = 1$$

Hence, the values of a and b are 2 and 1 respectively for which f is a continuous function.

31.

Show that the function defined by $\mathbf{f(x) = \cos(x^2)}$ is a continuous function

Ans - The given function is $f(x) = \begin{cases} 5, \text{if } x \leq 2 \\ ax + b, \text{if } 2 < x < 10 \\ 21, \text{if } x \geq 10 \end{cases}$

Function f is defined at every point on the real number line.
i.e. if the function f is continuous then f is continuous at every real number.

So let f be continuous at $x = 2$ and $x = 10$.

$g(c)=\cos c$

Substitute $x = c + h$ into the function g.

When, $x \to c$, then $h \to 0$.

Then we have,

$$\lim_{x \to c} g(x) = \lim_{x \to c} \cos x$$

$$= \lim_{h \to 0} \cos(c + h)$$

$$= \lim_{h \to 0} [\cos c. \cos h - \sin c. \sin h]$$

$$= \lim_{h \to 0} (\cos c. \cos h) - = \lim_{h \to 0} (\sin c. \sin h)$$

$$= \cos c \cdot \cos 0 - \sin c \cdot \sin 0$$

$$= \cos c \times 1 - \sin c \times 0$$

$$= \cos c$$

$$\Rightarrow \lim_{x \to c} g(x) = g(c)$$

$\therefore$ Function $g(x) = \cos x$ is continuous.

Again, $h(x) = x^2$ is defined for every real point.

So, let consider k be a real number, then $h(k) = k^2$ and

$\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$

$\Rightarrow \lim_{x \to k} h(x) = h(k).$

∴ Function h is continuous

Now, remember that for real valued functions g and h, such that (g o h) is defined at c, if g is continuous at c and f is continuous at g(c), then (f o h) is continuous at c.

Hence, the function $f(x) = (g \circ h)(x) = \cos(x)$ is continuous.

32.

Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

Ans - The given function is $f(x) = |\cos x|$

Function f is defined for all real numbers. So function f can be expressed as the composition of two functions as,
$f = g \circ h$, where $g(x) = [x]$ and $h(x) = \cos x$

$[\because (goh)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)]$

It is to be proved that functions $g(x) = [x]$ and $h(x) = \cos x$ are continuous.

Remember that, $g(x) = |x|$, can be written as

$$g(x) = \begin{cases} -x, \text{if } x < 0 \\ x, \text{if} x \geq 0 \end{cases}$$

Since function g is defined for every real number, let us consider c be a real number.

Then there will be 3 cases, either $c < 0$, or $c > 0$, or $c = 0$.

Case (i), When $c < 0$

$g(c) = -c$

$\lim_{x \to c} g(x) = \lim_{x \to c} (-x) = -c$

$\Rightarrow \lim_{x \to c} g(x) = g(c)$

$\therefore$ Function g is continuous at every point x, for $x < 0$.

Case (ii), When $c > 0$

$g(c) = c$

$\lim_{x \to c} g(x) = \lim_{x \to c} x = c$

$\Rightarrow \lim_{x \to c} g(x) = g(c)$

$\therefore$ Function g is continuous at every point x for $x > 0$.

Case (iii), When $c = 0$

$g(c) = g(0) = 0$

Left-hand-limit of the function g at $x = 0$ is

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$

Right-hand-limit is

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$

$$\Rightarrow \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = g(0)$$

$\therefore$ Function g is continuous at $x = 0$.

By observing the above three discussions, we can conclude that the function g is continuous at every real point.

Now, since the function $h(x) = \cos x$ is defined for all real numbers, let's consider c be a real number. Then, substitute $x = c + h$ into the function h.

33.

Examine that $\sin|x|$ is a continuous function

Ans - First suppose that, $f(x) = \sin|x|$

Function f is defined for all real numbers and so f can be expressed as the composition of functions as, $f = g \circ h$, where $g(x) = \sin x$ and $h(x) = |x|$.

$[(g \circ h)(x) = g(h(x)) = g(|x|) = \sin|x| = f(x)]$

It is to be proved that functions $g(x) = \sin x$ and $h(x) = |x|$ are continuous.

Function $h(x) = |x|$ can be written as

$$h(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \text{ if } x \geq 0 \end{cases}$$

Function h is defined for every real number, and so let us consider c be a real number

Then, there will be 3 cases, either $c < 0$, or $c > 0$, or $c = 0$.

Case (i), When $c < 0$.

$h(c) = -c$

$$\lim_{x \to c} (-x) = \lim_{x \to c} x = -c$$

Therefore, $\lim_{x \to c} h(x) = h(c)$

$\therefore$ Function h is continuous at every point x for $x < 0$.

Case (ii), When $c > 0$

$h(c) = c$

$$\lim_{x \to c} (-x) = \lim_{x \to c} x = c$$

$$\Rightarrow \lim_{x \to c} h(x) -= h(c)$$

$\therefore$ Function h is continuous at every point x for $x > 0$.

Case (iii), When c = 0.

h(c) = h(0) = 0.

Left-hand-limit of the function h at x = 0 is

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} (-x) = 0$$

Right-hand -limit of the function h at x = 0 is

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} (x) = 0$$

$$\Rightarrow \lim_{x\to 0^-} h(x) = \lim_{x\to 0^+} (x) = h(0)$$

∴ Function h is continuous at x = 0.

By observing the above three discussions, we can conclude that the function h is continuous at every point.

Since function g (x) = sin x is defined for all real numbers, so let's consider c be a real number and substitute x = c + k into the function.

Now, when x → c and k → 0. Then, we have

$$g(c) = \sin c$$

$$\lim_{x\to c} g(x) = , \lim_{x\to c} \sin x$$

$$= \lim_{k\to 0} \sin(c + k)$$

$$= \lim_{k\to 0} [\sin c \cdot \cos k + \cos c \cdot \sin k]$$

$$= \lim_{k\to 0} (\sin c \cdot \cos k) + \lim_{k\to 0} (\cos c \cdot \sin k)$$

$$= \sin c \cdot \cos 0 + \cos c \cdot \sin 0$$

$$= \sin c + 0$$

$$= \sin c$$

$$\Rightarrow \lim_{x\to c} g(x) = g(c).$$

Hence, the function g is continuous.

34.

Find all the points of discontinuity of f defined by $\mathbf{f(x) = |x| - |x+1|}$.

Ans - The given function is $f(x) = |x| - |x+1|$. Let us consider two functions

$g(x) = |x|$ and $h(x) = |x+1|$

$\Rightarrow f = g - h$

Function $g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, \text{if } x < 0 \\ x, \text{if } x \geq 0 \end{cases}$$

Function g is defined for every real number and so let's consider c be a real number.

Then there will be 3 cases, either $c < 0$, or $c > 0$, or $c = 0$.

Case (i), When $c < 0$.

$g(c) = g(0) = -c$

$\lim_{x \to c} g(x) = \lim (-x) = -c$

$\Rightarrow \lim_{x \to c} g(x) = g(c)$

$\therefore$ Function g is continuous at every point x for $x < 0$

Case (ii), When $c > 0$.

$g(c) = c$

$\lim_{x \to c} g(x) = \lim_{x \to c} x = c$

$\Rightarrow \lim_{x \to c} g(x) = g(c)$.

$\therefore$ Function g is continuous at every point x for $x > 0$.

Case (iii), When $c = 0$.

$g(c) = g(0) = -0$

Left- hand-limit of the function g at $x = 0$ is

$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$

Right-hand-limit of the function g at $x = 0$ is

$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$.

$\Rightarrow \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$

$\therefore$ Function g is continuous at $x = 0$

Thus, we can conclude by observing the above three discussions that g is continuous at every real point.

Function $h(x) = |x+1|$ can be written as

$$h(x) = \begin{cases} -x(x+1), \text{if } x < -1 \\ x+1, \text{if, } x \geq -1 \end{cases}$$

Note that, the function h is defined for all real numbers, and so let consider c be a real number.

Case (i), When $c < -1$

$$h(c) = -(c+1)$$

$$\lim_{x \to c} [-(x+1)] = -(c+1)$$

$$\Rightarrow \lim_{x \to c} h(x) = h(c)$$

∴ Function h is continuous at every real point x for $x < -1$.

Case (ii), When $c > -1$

$$h(c) = c+1$$

$$\lim_{x \to c} h(x) = \lim_{x \to c} (x+1) = (c+1)$$

$$\Rightarrow \lim_{x \to c} h(x) = h(c)$$

∴ Function h is continuous at every real point x for > -1.

Case (iii), When $c = -1$

$h(c) = h(-1) = -1 + 1 = 0$

Left-hand-limit of the function h at $x = 1$ is

$$\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} [-(x + 1)] = -(-1 + 1) = 0$$

Right-hand-limit of the function h at $x = 1$ is

$$\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} (x + 1) = (-1 + 1) = 0$$

$$\Rightarrow \lim_{x \to 1^-} h = \lim_{x \to 1^+} h(x) = h(-1)$$

∴ Function h is continuous at $x = -1$.

Hence, by observing the above three discussions, we can conclude that function h is continuous for every real point. Since functions g and h are both continuous, so the function $f = g - h$ is also continuous. Hence, function f does not have any discontinuity points.

Exercise 5.2

1.

Differentiate the function with respect to x in
$\sin(x^2 + 5)$

Ans - Let $f(x) = \sin(x^2 + 5)$
$u(x) = x^2 + 5$
$v(t) = \sin t$

Then, $(vou)(x) = v(u(x)) = v(x^2 + 5)$

$= \tan(x^2 + 5) = f(x)$

$\Rightarrow$ f is a composite of two functions.

Put $t = u(a) = x^2 + 5$

$\Rightarrow \frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(x^2 + 5)$

$\frac{dt}{dx} = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2) + \frac{d}{dx}(5) = 2x + 0 = 2x$

$\therefore$ By chain rule. $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(x^2 + 5) \times 2x$

$= 2x \cos(x^2 + 5)$

2.

Differentiate the function with respect of x in cos(sinx)

Ans - Let $f(x) = \cos(\sin)$, $u(x) = \sin x$, and $v(t) = \cos t$

Then, $(vou)(x) = v(u(x)) = v(\sin x) = \cos(\sin x) = f(x)$

$\Rightarrow$ f is a composite function of two functions

Put $t = u(x) = \sin x$

$\therefore \frac{dv}{dt} = \frac{d}{dt}[\cos t] = -\sin t = -\sin(\sin x)$

$\frac{dt}{dx} = \frac{d}{dx}(\sin x) = \cos x$

$\therefore$ By chain rule, $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = -\sin(\sin x) . \cos x$

$= -\cos x \sin(\sin x)$

3.

Differentiate the function with respect of x in $\sin(ax + b)$

Ans - Let $f(x) = \sin(ax+b), u(x) = ax + b$, and $v(t) = \sin t$

Then,
$(vow)(x) = v\big(u(x)\big) = v(ax+b) = \sin(ax+b) = f(x)$

$\Rightarrow$ f is a composite function of two functions u and v.

Put $t = u(x) = ax + b$.

$\Rightarrow \frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax+b)$

$$\frac{dt}{dx} = \frac{d}{dx}(ax+b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

$\therefore$ By chain rule, we obtain

$$\frac{df}{dx} = \frac{dv}{dt}.\frac{dt}{dx} = \cos(ax+b).a$$

$= a\cos(ax+b)$

4.

Differentiate the function with respect of x in $\sec(\tan(\sqrt{x}))$

Ans - Let $f(x) = \sec\left(\tan(\sqrt{x})\right), u(x) = \sqrt{x},$

$v(t) = \tan t$, and $w(s) = \sec s$

Then, $(vou)(x) = w[v(u(x))] = w[v(\sqrt{x})]$
$= w(\tan\sqrt{x}) = \sec(\tan\sqrt{x}) = f(x)$

⇒ f is a composite function of three functions, u, v and w.

Put $s = v(t) = \tan t$ and $t = u(x) = \sqrt{x}$

$[s = \tan t]$

Then, $\frac{dw}{ds} = \frac{d}{ds}(\sec s = \sec s \tan s)$

$= \sec(\tan t) \cdot \tan(\tan t)$

$= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x})$ $[t = \sqrt{x}]$

$$\frac{ds}{dt} = \frac{d}{dt}(\tan t) \sec^2 t = \sec^2 \sqrt{x}$$

$$\frac{dt}{dx} = \frac{d}{dx}\sqrt{x} = \frac{d}{dx}(x^{\frac{1}{2}})$$

$$= \frac{1}{2} . x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

∴ By Chain rule, we obtain $\frac{dt}{dx} = \frac{dw}{ds} . \frac{ds}{dt} . \frac{dt}{dx}$

$$= \sec(\tan\sqrt{x}) \cdot \tan(\tan\sqrt{x}) \times \sec^2\sqrt{x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} \sec^2\sqrt{x} \ (\tan\sqrt{x}) \tan(\tan\sqrt{x})$$

$$= \frac{\sec^2\sqrt{x} \ \sec(\tan\sqrt{x})\tan(\tan\sqrt{x})}{2\sqrt{x}}$$

5.

Differentiate the function with respect of x in $\frac{\sin(ax+b)}{\cos(cx+d)}$

Ans - The given function is

$$f(x) = \frac{\sin(ax+b)}{\cos(cx+d)} = \frac{g(x)}{h(x)}, \text{where } g(x) = \sin(ax+b) \text{ and}$$
$$h(x) = \cos(cx+d)$$

$\therefore f = \frac{g'h - gh'}{h^2}$

Consider $g(x) = \sin(ax + b)$

Let $u(x) = ax + b, v(t) = \sin t$

Then
$(vou)(x) = v(u(x)) = v(ax + b) = \sin(ax + b) = g(x)$

$\therefore$ g is a composite function of two functions, u and v

Put $t = u(x) = ax + b$

$$\frac{dv}{dt} = \frac{d}{dt}(\sin t) = \cos t = \cos(ax + b)$$

$$\frac{dt}{dx} = \frac{d}{dx}(ax + b) = \frac{d}{dx}(ax) + \frac{d}{dx}(b) = a + 0 = a$$

$\therefore$ by chain rule, we obtain

$$g' = \frac{dg}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos(ax + b). a = a\cos(ax + b)$$

Consider $h(x) = \cos(cx + d)$

Let $p(x) = cx + d, q(y) = \cos y$

Then,
$(qop)(x) = q\big(p(x)\big) = q(cx + d) = \cos(cx + d) = h(x)$

$\therefore$ h is a composite function of two functions, p and q.

Put $y = p(x) = cx + d$

$$\frac{dq}{dy} = \frac{d}{dy}(\cos y) = -\sin y = -\sin(cx + d)$$

$$\frac{dy}{dx} = \frac{d}{dx}(cx + d) = \frac{d}{dx}(cx) + \frac{d}{dx}(d) = c$$

Therefore, by chain rule, we obtain $h' = \frac{dh}{dx} = \frac{dq}{dy}.\frac{dy}{dx} =$
$-\sin(cx + d)xc = -c\sin(cs + d)$

$\therefore f'$
$$= \frac{a\cos(ax + b).\cos(cx + d) - \sin(ax + b)\{-c\sin cx + d\}}{[\cos(cx + d)]^2}$$

$$= a\cos(ax + b)\sec(cx + d) + c\sin(ax + b)\tan(cx + d)\sec(cx + d)$$

6.

Differentiate the function with respect to x in
$\mathbf{\cos x^3 . \sin^2 (x^5)}$

Ans - $\cos x^3 . \sin^2(x^5)$

$$\frac{d}{dx}[\cos x^3 . \sin^2(x^5)]$$

$$= \sin^2(x^5) \times \frac{d}{dx}(\cos x^3)$$
$$+ \cos x^3 \times \frac{d}{dx}[\sin^2(x^5)]$$

$$= \sin^2(x^5) \times (-\sin x^3) \times \frac{d}{dx}(x^3)$$
$$+ \cos x^3 + 2\sin(x^5). \frac{d}{dx}[\sin x^5]$$

$$= \sin x^3 \sin^2(x^5) \times 3x^2 + 2\sin x^5 \cos x^5 . \cos x^5 \times \frac{d}{dx}(x^5)$$

$$= 3x^2 \sin x^3 . \sin^3(x^5) + 2\sin x^5 \cos x^5 \cos x^3 . x^5 x^4$$

$$= 10x^4 \sin x^5 \cos x^5 \cos x^3 - 3x^2 \sin x^3 \sin^2(x^5)$$

7.

Differentiate the function with respect to x in $2\sqrt{\cot(x^2)}$

Ans - $\frac{d}{dx}[2\sqrt{\cot(x^2)}]$

$$2.\frac{1}{2\sqrt{\cot(x^2)}}\times\frac{d}{dx}[\cot(x^2)]$$

$$=\sqrt{\frac{\sin(x^2)}{\cos(x^2)}}\times-\mathrm{cosec}^2\,(x^2)\times\frac{d}{dx}(x^2)$$

$$=\sqrt{\frac{\sin(x^2)}{\cos(x^2)}}\times\frac{1}{\sin^2(x^2)}x(2x)$$

$$=\frac{-2x}{\sqrt{\cos x^2\sqrt{\sin x^2\sin x^2}}}$$

$$\frac{-2\sqrt{2x}}{\sqrt{2\sin x^2\cos x^2\sin x^2}}$$

$$=\frac{-2\sqrt{2x}}{\sin x^2\sqrt{\sin 2x^2}}$$

8.

Differentiate the function with respect to x in $\cos(\sqrt{x})$

Ans - Let $f(x) = \cos(\sqrt{x}), u(x) = \sqrt{x}$ and , $v(t) = \cos t$

Then, $(vou)(x) = v(u(x))$

$= v(\sqrt{x}) = \cos\sqrt{x} = f(x)$

$\Rightarrow$ f is a composite function of two functions, u and v, such that $t = u(a) = \sqrt{x}$.

Then,

$$\frac{dt}{dx} = \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}})$$

$$\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{dv}{dx} = \frac{d}{dx}(\cos t) = -\sin t = -\sin\sqrt{x}$$

$\therefore$ By using chain rule, we obtain $\frac{dt}{dx} = \frac{dv}{dt}.\frac{dt}{dx}$

$$= \sin(\sqrt{x}).\frac{1}{2\sqrt{x}}$$

$$= -\frac{1}{2\sqrt{x}}\sin(\sqrt{x})$$

$$= -\frac{\sin\sqrt{x}}{2\sqrt{x}}$$

9.

Prove that the function f given by

$$\mathbf{f(x) = |x - 1|, x \in R}$$

is not differentiable at $\mathbf{x = 1}$.

Ans - The given function is $f(x) = |x - 1|, x \in R$

Function f is differentiable at a point $x = c$ in its domain if both

$\lim_{k \to 0^-} \frac{f(c+h) - f(c)}{h}$ and $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$ are equal.

To check the differentiability of function at $x = 1$, consider the left-hand limit of f at $x = 1$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{f|I+h-1| - |1-1|}{h}$$

$$= \lim_{h \to 0^-} \frac{|h| - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} \qquad (h < 0 \Rightarrow |h| = -h)$$

$$= -1$$

Consider the right-hand limit of f at $x = 1$

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{f|I+h-1| - |1-1|}{h}$$

$$= \lim_{h \to 0^+} \frac{|h| - 0}{h} = \lim_{h \to 0^+} \frac{h}{h} \qquad (h > 0 \Rightarrow |h| = h)$$

$$= 1$$

Since the left and right hand limits of f at $x = 1$ are not equal, f is not differentiable at $x = 1$.

10.

Prove that the greatest integer function defined by

$f(x) = [x], 0 < x < 3$

is not differentiable $x = 1$ and $x = 2$.

Ans - The given function f is $f = (x) = [x], 0 < x < 3$

Function f is differentiable at a point $x = c$ in its domain if both

$\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h}$ and $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$ are finite and equal.

To check the differentiable of function at $x = 1$, consider the left hand limit of f at $x = 1$.

$$\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^-} \frac{f[1+h]-[1]}{h}$$

$$\lim_{h \to 0^-} \frac{0+1}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

Consider the right-hand limit of f at $x = 1$

$$\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^+} \frac{f[1+h]-[1]}{h}$$

$$= \lim_{h \to 0^+} \frac{1-1}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right limits of f at $x = 1$ are not equal, f is not differentiable at $x = 1$

To check the differentiable of function at $x = 2$, consider the left-hand limit of f at $x = 2$.

$$\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^-} \frac{[2+h] - [2]}{h}$$

$$\lim_{h \to 0^-} \frac{1-2}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

Consider the right-hand limit of f at $x = 2$

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0^+} \frac{[2+h] - [2]}{h}$$

$$\lim_{h \to 0^-} \frac{1-2}{h} = \lim_{h \to 0^-} 0 = 0$$

Since the left and right hand limits of f at $x = 2$ are not equal, f is not differentiable at $x = 2$.

Exercise 5.3

1.

Find $\frac{dy}{dx}$ in $2x + 3y = \sin x$

Ans - Given relationship is $2x + 3y = \sin x$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dy}(2x + 3y) = \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dy}(3y) = \cos x$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$$

$$\Rightarrow 3\frac{dy}{dx} = \cos x$$

$$\therefore \frac{dx}{dy} = \frac{\cos x - 2}{3}$$

2.

Find $\frac{dy}{dx}$ in $2x + 3y = \sin y$

Ans - Given relationship is $2x + 3y = \sin y$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin y)$$

$$\Rightarrow 2 + 3\frac{dy}{dx} = \cos y\ \frac{dy}{dx} \text{[By using chain rule]}$$

$$\Rightarrow 3 = (\cos y - 3)\ \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2}{\cos y - 3}$$

3.

Find $\frac{dy}{dx}$ in $ax + by^2 = \cos y$

Ans - Given relationship is $ax + by^2 = \cos y$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y)$$

$$\Rightarrow a + b\frac{d}{dx}(y^2) = \frac{d}{dx}(\cos y)$$

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx} \text{ and } \frac{d}{dx}(\cos y) = -\sin y\frac{dy}{dx}$$

Using the chain rule we get,

$$\Rightarrow a + b \times 2y\frac{dy}{dx} = -\sin y\frac{dy}{dx}$$

$$\Rightarrow (2by + \sin y)\frac{dy}{dx} = -a$$

$$\therefore \frac{dy}{dx} = \frac{-a}{2by + \sin y}$$

4.

Find $\frac{dy}{dx}$ in $xy + y^2 = \tan x + y$

Ans - Given relationship is $xy + y^2 = \tan x + y$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dx}(xy + y^2) = \frac{d}{dx}(\tan x + y) = \frac{d}{dx}(\cos y)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y.\frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

Using product rule and chain rule we get,

$$\Rightarrow y.1 + x\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x + 2y - 1)}$$

5.

Find $\frac{dy}{dx}$ in $x^2 + xy + y^2 = 100$

Ans - Given relationship is $x^2 + xy + y^2 = 100$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(100)$$

Derivative of the constant function is 0

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) = \frac{d}{dx}(y^2) = 0$$

$$\Rightarrow 2x + \left[y.\frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0$$

Using product rule and chain rule we get,

$$\Rightarrow 2x + y.1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y + (x + 2y)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x + y}{x + 2y}$$

6.

Find $\frac{dy}{dx}$ in $x^3 + x^2y + xy^2 + y^3 = 81$

Ans - Given relationship is $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dx}(x^3 + x^2y + xy^2 + y^3) = \frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}(y^3) = 0$$

$$\Rightarrow 3x^2 + \left[y\frac{d}{dx}(x^2) + x^2\frac{dy}{dx}\right] + \left[y^2\frac{d}{dx}(x) + x\frac{d}{dx}(y^2)\right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + \left[y.\,2x + x^2\frac{dy}{dx}\right] + \left[y^2\frac{d}{dx}(x) + x\frac{d}{dx}(y^2)\right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow (x^2 + 2xy + 3y^2)\frac{dy}{dx}(3x^2 + 2xy + y^2)$$

$$\therefore \frac{dy}{dx} = \frac{-(3x^2 + 2xy + y^2)}{(x^2 + 2xy + 3y^2)}$$

7.

Find $\frac{dy}{dx}$ in $\sin^2 y + \cos xy = k$

Ans - Given relationship is $\sin^2 y + \cos xy = k$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dx}(\sin^2 y + \cos xy) = \frac{d}{dx}(k)$$

$$\Rightarrow \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) = 0$$

Using the chain rule we get,

$$\frac{d}{dx}(\sin^2 y) = 2\sin y \frac{d}{dx}(\sin y) = 2\sin y \cos y \frac{dy}{dx}$$

$$\frac{d}{dx}(\cos xy) = -\sin xy \frac{d}{dx}(xy) = -\sin xy\, [y\frac{d}{dx}(x) + x\frac{dy}{dx}]$$

$$= -\sin xy \left[y.1 + x\frac{dy}{dx}\right] = -y\sin xy - x\sin xy \frac{dy}{dx}$$

From the above equations we get,

$$2\sin y \cos y \frac{dy}{dx} - y\sin xy - x\sin xy \frac{dy}{dx} = 0$$

$$\Rightarrow (2\sin y \cos y - x\sin xy)\frac{dy}{dx} - y\sin xy$$

$$\Rightarrow (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

8.

Find $\frac{dy}{dx}$ in $\sin^2 x + \cos^2 y = 1$

Ans - Given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating with respect to x we get,

$$\Rightarrow \frac{dy}{dx}(\sin^2 x + \cos^2 y) = \frac{dy}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}(\sin^2 x) + \frac{d}{dx}(\cos^2 y) = 0$$

$$\Rightarrow 2\sin x \cdot \frac{d}{dx}(\sin x) + 2\cos y \cdot \frac{d}{dx}(\cos y) = 0$$

$$\Rightarrow 2\sin x \cos x + 2\cos y(-\sin y).\frac{dy}{dx} = 0$$

$$\sin 2x - \sin 2y\frac{dy}{dx} = 0$$

$$\frac{dx}{dy} = \frac{\sin 2x}{\sin 2y}$$

9.

Find $\frac{dy}{dx}$ in $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Ans - Given relationship is $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$\Rightarrow \sin y = \frac{2x}{1+x^2}$$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \cos y \frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

The function $\frac{2x}{1+x^2}$ is of the form of $\frac{u}{v}$. Thus, by quotient rule, we get

$$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}(2x) - 2x.\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2).2 - 2x[0+2x]}{(1+x^2)^2} = \frac{2+2x^2-4x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

Given $\sin y = \frac{2x}{1+x^2}$

$$\Rightarrow \cos y = \sqrt{1-\sin^2 y}$$

$$= \sqrt{1-\left(\frac{2x}{1+x^2}\right)^2} = \sqrt{\frac{(1+x^2)^2-4x^2}{(1+x^2)^2}}$$

$$= \sqrt{\frac{(1+x^2)^2}{(1+x^2)^2}} = \frac{1-x^2}{1+x^2}$$

From above equation, we get

$$\Rightarrow \frac{1-x^2}{1+x^2} \times \frac{dy}{dx} = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

10.

Find $\frac{dy}{dx}$ in $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Ans - Given relationship is $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

Putting $x = \tan\theta$

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - \tan^2\theta}\right)$$

$$y = \tan^{-1}(\tan 3\theta)\left(\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$y = 3\theta$$

Differentiating both sides w.r.t. x we get,

$$\frac{d(y)}{dx} = \frac{d(3\tan^{-1}x)}{dx}$$

$$\frac{dy}{dx} = 3\frac{d(\tan^{-1}x)}{dx}$$

$$\frac{dy}{dx} = 3\left(\frac{1}{1 + x^2}\right) \qquad \left((\tan^{-1}x)' = \frac{1}{1 + x^2}\right)$$

$$\frac{dy}{dx} = \frac{3}{1 + x^2}$$

11.

Find $\frac{dy}{dx}$ in $y = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$, $0 < x < 1$

Ans - Given relationship is $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow \cos y = \frac{1-x^2}{1+x^2}$

$$\Rightarrow \frac{1 - \tan^2 \frac{y}{2}}{1 + \tan^2 \frac{y}{2}} = \frac{1 - x^2}{1 + x^2}$$

On comparing L.H.S. and R.H.S. we get,

$$\Rightarrow \tan \frac{y}{2} = x$$

$y = 2(\tan^{-1} x)$

Differentiating with respect to x, we get

$$\Rightarrow \frac{dy}{dx} = \frac{d(2\tan^{-1} x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 2\frac{d(\tan^{-1} x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 2\left(\frac{1}{1 + x^2}\right)$$

$$\therefore \frac{dy}{dx} = \frac{2}{1 + x^2}$$

12.

Ans - Given relationship is $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$$

$$\Rightarrow (1+x^2)\sin y = 1 - x^2$$

$$\Rightarrow (1+\sin y)x^2 = 1 - \sin y$$

$$\Rightarrow x^2 = \frac{1-\sin y}{1+\sin y}$$

$$\Rightarrow x^2 = \frac{\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2} + \sin\frac{y}{2}\right)^2}$$

$$\Rightarrow x = \frac{\cos\frac{y}{2} - \sin\frac{y}{2}}{\cos\frac{y}{2} + \sin\frac{y}{2}}$$

$$\Rightarrow x = \frac{1 - \tan\frac{y}{2}}{1 + \tan\frac{y}{2}}$$

$$\Rightarrow x = \tan(\frac{\pi}{4} - \frac{\pi}{2})$$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}[\tan\left(\frac{\pi}{4} - \frac{y}{2}\right)]$$

$$\Rightarrow 1 = \sec^2\left(\frac{\pi}{4} - \frac{y}{2}\right).\frac{d}{dt}\left(\frac{\pi}{4} - \frac{y}{2}\right)$$

$$\Rightarrow 1 = \left[1 + \tan^2\left(\frac{\pi}{4} - \frac{y}{2}\right)\left(-\frac{1}{2}\frac{dy}{dx}\right)\right]$$

$$\Rightarrow 1 = (1+x^2)\left(-\frac{1}{2}\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{1+x^2}$$

13.

Find $\frac{dy}{dx}$ in $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$

Ans - Given relationship is $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow \cos y = \frac{2x}{1+x^2}$

Differentiating with respect to x we get,

$$\Rightarrow \frac{d}{dx}(\cos y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \frac{(1+x^2).\frac{d}{dx}(2x) - 2x.\frac{d}{dt}(1+x^2)}{(1+x^2)^2}$$

$$\Rightarrow -\sqrt{1-\cos^2 y}\frac{dy}{dx} = \frac{(1+x^2)\times 2 - 2x.2x}{(1+x^2)^2}$$

$$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx} = -\left[\frac{2(1-x^2)}{(1+x^2)^2}\right]$$

$$= \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}}\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}\cdot\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{1+x^2}$$

14.

Find $\frac{dy}{dx}$ in $y = \sin^{-1}(2x\sqrt{1-x^2}), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Ans – Given relationship is $y = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow \sin y = 2x\sqrt{1-x^2}$

Differentiating with respect to x we get,

$$\cos y \frac{dy}{dx} = 2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{dx}{dx}\right]$$

$$= \sqrt{1-\sin^2 y}\ \frac{dy}{dx} = 2\left[\frac{x}{2}.\frac{-2x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right]$$

$$= \sqrt{1-(2x\sqrt{1-x^2})^2}\,\frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$$

$$= \sqrt{1-4x^2(1-x^2)}\,\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$= \sqrt{(1-2x^2)^2}\,\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$= (1-2x^2)\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

15.

Find $\frac{dy}{dx}$ in $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Ans - Given relationship is $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$$\Rightarrow \sec y = \frac{1}{2x^2 - 1}$$

$$\Rightarrow \cos y = 2x^2 - 1$$

$$\Rightarrow 2x^2 = 1 + \cos y$$

$$\Rightarrow 2x^2 = 2\cos^2\frac{y}{2}$$

Differentiating with respect to x we get,

$$\Rightarrow \frac{dy}{dx}(x) = \frac{d}{dx}\left(\cos\frac{y}{2}\right)$$

$$\Rightarrow 1 = -\sin\frac{y}{2} \cdot \frac{d}{dx}\left(\frac{y}{2}\right)$$

$$\Rightarrow -\frac{1}{\sin\frac{y}{2}} = \frac{1}{2}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{\sin\frac{y}{2}} = \frac{-2}{\sqrt{1 - \cos^2\frac{y}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sqrt{1 - x^2}}$$

Exercise 5.4

1.

Differentiating the following w.r.t x: $\frac{e^x}{\sin x}$

Ans - Let $y = \frac{e^x}{\sin x}$

Now differentiating w.r.t x we get

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x \times \frac{d}{dx}(e^x) - e^x \times \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x(\sin x - \cos x)}{\sin^2 x}, x \neq n\pi, n \in \mathbb{Z}$$

2.

Differentiating the following w.r.t x: $e^{\sin^{-1} x}$

Ans - Let $y = e^{\sin^{-1} x}$

Now differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1} x}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} x} \times \frac{d}{dx}(\sin^{-1} x)$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} x} \times \frac{1}{\sqrt{1 - x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}, x \in (-1, 1)$$

3.

Differentiating the following w.r.t. x: e^{x^3}

Ans - Let $y = e^{x^3}$

By using the quotient rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x^3}\right)$$

$$\Rightarrow \frac{dy}{dx} = e^{x^3} \times \frac{d}{dx}(x^3)$$

$$\Rightarrow \frac{dy}{dx} = e^{x^3} \times 3x^2$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 e^{x^3}$$

4.

Differentiating the following w.r.t. x: $\sin(\tan^{-1} e^{-x})$

Ans - Let $y = \sin(\tan^{-1} e^{-x})$

By using chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\tan^{-1} e^{-x})]$$

$$\Rightarrow \frac{dy}{dx} = \cos(\tan^{-1} e^{-x}) \times \frac{d}{dx}(\tan^{-1} e^{-x})$$

$$\Rightarrow \frac{dy}{dx} = \cos(\tan^{-1} e^{-x}) \times \frac{1}{1 + (e^{-x})^2} \frac{d}{dx}(e^{-x})$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos(\tan^{-1} e^{-x})}{1 + (e^{-x})^2} \times e^{-x} \times \frac{d}{dx}(-x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{-x}\cos(\tan^{-1} e^{-x})}{1 + e^{-2x}} \times (-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^{-x}\cos(\tan^{-1} e^{-x})}{1 + e^{-2x}}$$

5.

Differentiating the following w.r.t. x: $\log(\cos e^x)$

Ans – Let $y = \log(\cos e^x)$

By using chain rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\cos e^x)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos e^x} \times \frac{d}{dx}(\cos e^x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos e^x} \times \frac{d}{dx}(\cos e^x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos e^x} \times (-\sin e^x) \times \frac{d}{dx}(e^x)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sin e^x}{\cos e^x} \times (e^x)$$

$$\Rightarrow \frac{dy}{dx} = -e^x \tan e^x, e^x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{N}$$

6.

Differentiating the following w.r.t. x: $e^x + e^{x^2} + \cdots \ldots e^{x^5}$

Ans - Let $y = e^x + e^{x^2} + \cdots \ldots e^{x^5}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(e^x) + \frac{d}{dx}\left(e^{x^2}\right) + \frac{d}{dx}\left(e^{x^3}\right) + \frac{d}{dx}\left(e^{x^4}\right) +$$

$$\Rightarrow \frac{dy}{dx} = e^x + \left[e^{x^2}\frac{d}{dx}(x^2)\right] + \left[e^{x^3}\frac{d}{dx}(x^3)\right] + \left[e^{x^4}\frac{d}{dx}(x^4)\right] + \left[e^{x^5}\frac{d}{dx}(x^5)\right]$$

$$\Rightarrow \frac{dy}{dx} = e^x\left[e^{x^2} \times 2x\right] + \left[e^{x^3} \times 3x^2\right] + \left[e^{x^4} \times 4x^3\right] + \left[e^{x^5} \times 5x^4\right]$$

$$\Rightarrow \frac{dy}{dx} = e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^5}$$

7.

Differentiation the following w.r.t: $\sqrt{e^{\sqrt{x}}}, x > 0$

Ans – Let $y = \sqrt{e^{\sqrt{x}}}$

Then $y^2 = e^{\sqrt{x}}$

By differentiating the above equation with respect to x, we get

$y^2 = e^{\sqrt{x}}$

$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}}\frac{d}{dx}(\sqrt{x})$ (By applying the chain rule)

$\Rightarrow 2y\frac{dy}{dx} = e^{\sqrt{x}} \times \frac{1}{2} \times \frac{1}{\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4y\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\sqrt{x}}, x > 0$

8.

Differentiating the folllowing w.r.t x: $\log(\log x), x > 1$

Ans - Let $y = \log(\log x)$

By using chain rule, in the above equation, we get

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\log x} \times \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\log x} \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x\log x}, x > 1$$

9.

Differentiating the following w.r.t x: $\frac{\cos x}{\log x}, x > 0$

Ans - Let $y = \frac{\cos x}{\log x}$

By using the quotient rule, we obtain

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x) \times \log x - \cos x \times \frac{d}{dx}(\log x)}{(\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(-\sin x)\log x - \cos x \times \frac{1}{x}}{(\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x\log x \times \sin x + \cos x}{x(\log x)^2}, x > 0$$

10.

Differentiating the following w.r.t x: $\cos(\log x + e^x), x > 0$

Ans - Let $y = \cos(\log x + e^x)$

BY using the chain rule in the above equation, we get

$y = \cos(\log x + e^x)$

$$\frac{dy}{dx} = -\sin(\log x + e^x) \times \frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x) \times [\frac{d}{dx}(\log x) + \frac{d}{dx}(e^x)]$$

$$= -\sin(\log x + e^x) \times (\frac{1}{x} + e^x)$$

$$= -\left(\frac{1}{x} + e^x\right) \sin(\log x + e^x), x > 0$$

Exercise 5.5

1.

Differentiate the function given w.r.t. x.
$\mathbf{y = \cos x \times \cos 2x \times \cos 3x}$

Ans - Given function is $y = \cos x \times \cos 2x \times \cos 3x$

Taking logarithm on both sides we get,

$\log y = \log(\cos x \times \cos 2x \times \cos 3x)$

$\Rightarrow \log y = \log(\cos x) + \log(\cos 2x) + \log(\cos 3x)$, by the property of logarithm.

Now, differentiating both sides of the equation w.r.t. x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos x} \times \frac{d}{dx}(\cos x) + \frac{1}{\cos x} \times \frac{d}{dx}(\cos x) + \frac{1}{\cos 3x} \times \frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{\sin x}{\cos x} - \frac{\sin 2x}{\cos 2x} \times \frac{d}{dx}(2x) - \frac{\sin 3x}{\cos 3x} \times \frac{d}{dx}(3x)\right]$$

$$\frac{dy}{dx} = -\cos x \times \cos 2x \times \cos 3x[\tan x + 2\tan 2x + 3\tan 3x]$$

2.

Differentiate the function given w.r.t. x.

$$\mathbf{y} = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

Ans - Given function is $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$.

Taking logarithm on both sides of the equation give,

$$\log y = \log \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$$

$$\Rightarrow \log y = \frac{1}{2}\log\left[\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right]$$

$$\Rightarrow \log y = \frac{1}{2}[\log\{(x-1)(x-2)\} - \log\{(x-3)(x-4)(x-5)\}]$$

$$\Rightarrow \log y = \frac{1}{2}[\log(x-1 + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$$

Now, differentiating both sides of equation w.r.t. x give

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}[\log(x-1 + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1}\times\frac{d}{dx}(x-1) + \frac{1}{x-2}\times\frac{d}{dx}(x-2)\times\frac{1}{x-3}\times\frac{d}{dx}(x-3) \times\frac{1}{x-4}\times\frac{d}{dx}(x-4) - \frac{1}{x-5}\times\frac{d}{dx}(x-5)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}+\frac{1}{x-4}+\frac{1}{x-5}\right)$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3} + \frac{1}{x-4}+\frac{1}{x-5}\right]$$

3.

Differentiate the function given w.r.t. x.
$\mathbf{y = (\log x)^{\cos x}}$.

Ans - Given function is $y = (\log x)^{\cos x}$

Taking logarithm on both sides of the equation give,

$\log y = \cos x \,.\, \log(\log x)$

Now, differentiating both sides of equation w.r.t. x give,

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{d}{dx}(\cos x) \times \log(\log x) + \cos x \times \frac{d}{dx}[\log(\log x)]$$

$\Rightarrow \frac{1}{y} \times \frac{dy}{dx} = -\sin x \log(\log x) + \cos x \times \frac{1}{\log x} \times \frac{d}{dx}(\log x)$, by applying the chain rule.

$$\Rightarrow \frac{dy}{dx} = y\left[-\sin x \log(\log x) + \frac{\cos x}{\log x} \times \frac{1}{x}\right].$$

Therefore,

$$\frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x \times \log(\log x)\right]$$

4.

Differentiate the function given w.r.t. x.
$\mathbf{y = x^x - 2^{\sin x}}$

Ans - Given function is $y = x^x - 2^{\sin x}$.

Let $x^x = u$ ....(1)

And $2^{\sin x} = v$ ....(2)

$\Rightarrow y = u - v$ .... (3)

Differentiating equation (3) w.r.t. x gives

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad \ldots \ldots (4)$$

Taking logarithm on both sides of equation (1) we get,

$\log(u) = \log(x^x)$

$\Rightarrow \log u = x \log x$

Differentiating both sides of equation w.r.t. x we get,

$$\frac{1}{u}\frac{du}{dx} = \left[\frac{d}{dx}(x) \times \log x + x \times \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log x + x \times \frac{1}{x})\right]$$

$$\Rightarrow \frac{du}{dx} = x^x(\log x + 1)$$

$$\Rightarrow \frac{du}{dx} = x^x(1 + \log x) \qquad \dots\dots \quad (5)$$

Now, taking logarithm on both sides of equation (2) we get,

$\log(2^{\sin x}) = \log v$

$\Rightarrow \log v = \sin x \times \log 2$

Differentiating both sides of equation w.r.t. x we get,

$$\frac{1}{v} \times \frac{dv}{dx} = \log 2 \times \frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v \log 2 \cos x$$

$$\Rightarrow \frac{dv}{dx} = 2^{\sin x} \cos x \log 2 \dots\dots\dots\dots\dots. (6)$$

∴ From the equation (4), (5) and (6) we get,

$$\frac{dy}{dx} = x^x(1 + \log x) - 2^{\sin x} \cos x \log 2$$

5.

Differentiate the function given w.r.t. x.
$\mathbf{y = (x+3)^2(x+4)^3(x+5)^4}$

Ans - Given function is $y = (x+3)^2(x+4)^3(x+5)^4$.

Taking logarithm on both sides of equation w.r.t. x, give

$\log y - \log[(x+3)^2(x+4)^3(x+5)^4]$

$\Rightarrow \log y = 2\log(x+3) + 3\log(x+4)4\log(x+5)$

Now, differentiating both sides of equation w.r.t. x, give

$$\frac{1}{y} \times \frac{dy}{dx} = 2 \times \frac{1}{x-3} \times \frac{d}{dx}(x+3) + 3 \times \frac{1}{x+4} \times \frac{d}{dx}(x+4) + 4 \times \frac{1}{x+5} \times \frac{d}{dx}(x+5)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4 \times \left[\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4 \times \left[\frac{2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)}{(x+3)(x+4)(x+5)}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4 - [2(x^2+9x+20) + 3(x^2+9x+15) + 4(x^2+7x+12)]$$

$$\Rightarrow \frac{dy}{dx} = (x+3)(x+4)^2(x+5)^2(9x^2+70x+133)$$

6.

Differentiate the function given w.r.t. x.

$$\mathbf{y = \left(x + \frac{1}{x}\right)^2 + x^{\left(x+\frac{1}{x}\right)}}$$

Ans - Given function is $y = \left(x + \frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)}$.

Let $u = \left(x + \frac{1}{x}\right)^x$ and $v = x^{\left(x+\frac{1}{x}\right)}$

$\Rightarrow y = u + v \qquad .... (1)$

Differentiating equation (1) on both sides w.r.t. x give

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} = \frac{dv}{dx} \qquad ... (2)$$

Now, $u = \left(x + \frac{1}{x}\right)^x$

$$\Rightarrow \log u = \log\left(x + \frac{1}{x}\right)^x$$

$$\Rightarrow \log u = x \log x + \frac{1}{x}$$

Differentiating both sides of equation w.r.t. to x gives

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \log\left(x + \frac{1}{x}\right) + x \times \frac{d}{dx}\left[\log\left(x + \frac{1}{x}\right)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = 1 \times \log\left(x + \frac{1}{x}\right) + x \times \frac{1}{\left(x + \frac{1}{x}\right)} \times \frac{d}{dx}\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log\left(x + \frac{1}{x}\right) + \frac{x}{\left(x + \frac{1}{x}\right)} \times \left(x + \frac{1}{x^2}\right)\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[\log\left(x + \frac{1}{x}\right) + \frac{\left(x - \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[\log\left(x + \frac{1}{x}\right) + \frac{x^2 + 1}{x^2 - 1}\right]$$

$$\Rightarrow \frac{du}{dx} = \left(x + \frac{1}{x}\right)^x \left[\frac{x^2 + 1}{x^2 - 1} + \log\left(x + \frac{1}{x}\right)\right] \qquad \ldots (3)$$

Also $v = x^{\left(x + \frac{1}{x}\right)}$

$$\Rightarrow \log v = \log\left[x^{x^{\left(x + \frac{1}{x}\right)}}\right]$$

$$\Rightarrow \log v = \left(x + \frac{1}{x}\right)\log x$$

Differentiate both sides of equation w.r.t. x gives

$$\frac{1}{v}\times\frac{dv}{dx}=\left[\frac{d}{dx}\left(x+\frac{1}{x}\right)\right]\times\log x+\left(x+\frac{1}{x}\right)\times\frac{d}{dx}\log x$$

$$\Rightarrow\frac{1}{v}\times\frac{dv}{dx}=-\frac{\log x}{x^2}+\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow\frac{dv}{dx}=v\left[\frac{-\log x+x+1}{x^2}\right]\qquad \ldots\ldots(4)$$

Hence, from the equation (2), (3) and (4), give

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{2+1}+\log\left(x+\frac{1}{x}\right)\right]$$
$$+x^{\left(x+\frac{1}{x}\right)}\left(\frac{x+1-\log x}{x^2}\right)$$

7.

Differentiate the function given w.r.t. x.
$\mathbf{y=(\log x)^x+x^{\log x}}$

Ans - Given function is $y=(\log x)^x+x^{\log x}$

Let $u=(\log x)^x$ and $v=x^{\log x}$

$\Rightarrow y=u+v$

Differentiating both sides of equation w.r.t. x gives.

$$\Rightarrow\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}\qquad \ldots(1)$$

We know, $u=(\log x)^x$

$\Rightarrow \log u=\log[(\log x)^x]$

$\Rightarrow \log u=x\log(\log x)$

Differentiating both sides of equation w.r.t. x gives

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \log(\log x) + x \times \frac{d}{dx}\log(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log(\log x) + x \times \frac{1}{\log x} + \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x}\left[\log(\log x) + \frac{x}{\log x} \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x}\left[\log(\log x) + \frac{1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x}\left[\frac{\log(\log x) \times \log x + 1}{\log x}\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^{x-1}[1 + \log x \log(\log x)] \qquad \dots (2)$$

We know, $v = x^{\log x}$

$$\Rightarrow \log v = \log(x^{\log x})$$

$$\Rightarrow \log v = \log x \log x = (\log x)^2$$

Differentiating both sides of equation w.r.t. x gives

$$\frac{1}{v} = \frac{dx}{dx} = \frac{d}{dx}[(\log x)^2]$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x}\frac{\log x}{x}$$

$$\Rightarrow \frac{dv}{dx} = 2x^{\log x} \times \log x \qquad \dots . (3)$$

Hence, from the equations (1), (2) and (3), we get

$$\frac{dy}{dx} = (\log x)^{x+1}[1 + \log x \times \log(\log x)]5 + 2x^{\log x-1} \times \log x$$

8.

Differentiate the function given w.r.t. x.

$$\mathbf{y = (\sin x)^x + \sin\sqrt{x}}$$

Ans - Given function is $y = (\sin x)^x + \sin^{-1}\sqrt{x}$.

Let $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\Rightarrow y = u + v.$

Differentiating both sides of equation w.r.t. x gives.

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad \dots (1)$$

We know, $u = (\sin x)^x$

$\Rightarrow \log u = x \log(\sin x)^x$

$\Rightarrow \log u = x \log(\sin x)$

Differentiating both sides of equation w.r.t. x gives

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \log(\sin x) + x \times \frac{d}{dx}[\sin x]$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \log(\sin x) + x \times \frac{1}{\sin x} \times \frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left[\log(\sin x) + \frac{x}{\sin x} \times \cos x\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x (x \cot x + \log \sin x) \qquad \dots (2)$$

Again, $v = \sin^{-1}\sqrt{x}$

Differentiating both sides of equation w.r.t. x gives

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{2})^2}} \times \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x-x^2}}$$

Hence, from the equation (1), (2) and (3), gives

$$\frac{dv}{dx} = (\sin x)^2(x\cot x + \log\sin x) + \frac{1}{2\sqrt{x-x^2}}$$

9.

Differentiate the function given w.r.t. x.

$\mathbf{y = x^{\sin x} + (\sin x)^{\cos x}}$

Ans - Given function is $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $u = x^{\sin x}$ and $v = (\sin x)^{\cos x}$.

$\Rightarrow y = u + v$

Differentiating both sides of equation w.r.t. x gives

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad \dots (1)$$

We know, $u = x^{\sin x}$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\sin x) \times \log x + \sin x \times \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\cos x \log x + \sin x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x}\left[\cos x \log x + \frac{\sin x}{x}\right] \qquad \dots (2)$$

Again, $v = (\sin x)^{\cos x}$

$\Rightarrow \log v = \log(\sin x)^{\cos x}$

$\Rightarrow \log v = \cos x \log(\sin x)$

Differentiating both sides of equation w.r.t. x gives

$$\frac{1}{v}\frac{d}{dx} = \frac{d}{dx}(\cos x) \times \log(\sin x) + \cos x \times \frac{d}{dx}[\log(\sin x)]$$

$$\Rightarrow \frac{dv}{dx} = v\left[-\sin x \times \log(\sin x) + \cos x \times \frac{1}{\sin x} \times \frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^{\cos x}[-\sin x \log \sin x + \cot x \cos x]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}[\cos x \cot x + \sin x \log \sin x] \qquad \dots\dots (3)$$

Hence, from the equation (1), (2) and (3), we get

$$\frac{du}{dx} = x^{\sin x}\left(\cos x \log x + \frac{\sin x}{x}\right) + (\sin x)^{\cos x}[\cos x \cot x + \sin x \log \sin x]$$

10.

Differentiate the function given w.r.t. x.

$$\mathbf{y = x^{\cos x} + \frac{x^2+1}{x^2-1}}$$

Ans - Given function is $y = x^{x\cos x} + \frac{x^2+1}{x^2-1}$.

Let $u = x^{x\cos x}$ and $v = \frac{x^2+1}{x^2-1}$

$\Rightarrow y = u + v.$

Differentiating both sides of equation w.r.t. x gives

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad \dots (1)$$

We know, $u = x^{x\cos x}$

Differentiating both sides of equation w.r.t. x gives

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x) \times \cos x \log x + x \times \frac{d}{dx}(\cos x) \times \log x + x\cos x \times \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[1 \times \cos x \times \log x + x \times (-\sin x)\log x + x\cos x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^{x\cos x}(\cos x \log x - x \sin x \log x$$
$$+ \cos x) \qquad \dots (2)$$

Again, $v = \frac{x^2+1}{x^2-1}$

$$\Rightarrow \log v = \log(x^2 + 1) - \log(x^2 - 1)$$

Differentiating both sides of the equation with respect to x gives

$$\frac{1}{v} = \frac{dv}{dx} = \frac{2x}{x^2 + 1} - \frac{2x}{x^2 - 1}$$

$$\Rightarrow \frac{du}{dx} = \frac{x^2 + 1}{x^2 - 1} \times \left[\frac{-4x}{(x^2 + 1)(x^2 - 1)}\right]$$

$$\frac{dv}{dx} = \frac{-4x}{(x^2 - 1)^2} \qquad \dots\dots (3)$$

Hence, from the equation (1), (2) and (3), give

$$\frac{dv}{dx} = x^{x\cos x}[\cos x\,(1 + \log x) - x \sin x \log x] - \frac{-4x}{(x^2 - 1)^2}$$

11.

Differentiate the function given w.r.t. x.

$$\mathbf{y = (x\cos x)^x + (x\sin x)^{\frac{1}{x}}}$$

Ans - Given function is $y = (x\cos x)^x + (x\sin x)^{\frac{1}{x}}$.

Let $u = (x\cos x)^x$ and $v = (x\sin x)^{\frac{1}{x}}$

$\Rightarrow y = u + v.$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots.(1)$$

We know, $u = (\cos x)^x$

$\Rightarrow \log u = \log\ (x\cos x)^x$

$\Rightarrow \log u = x\log(x\cos x)$

$\Rightarrow \log u = x[\log x + \log\cos x]$

$\Rightarrow \log u = x\log x + x\log\cos x$

Differentiating both sides of equation w.r.t. x gives

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x\log x + x\log\cos x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\left\{\log x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\log x)\right\} + \left\{\log\cos x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\log\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[\left\{\log x \times 1 + x \times \frac{1}{x})\right\} + \left\{\log\cos x - 1 + x \times \frac{1}{\cos x} \times \frac{d}{dx}(\cos x)\right\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x\left[\{\log x + 1\} + \{\log\cos x - 1 + \frac{x}{\cos x} \times (-\sin x)\}\right]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x[(\log x + 1) + (\log\cos x - x\tan x)]$$

$$\Rightarrow \frac{du}{dx} = (x\cos x)^x[1 - x\tan x + (\log x + \log\cos x)]$$

$$\Rightarrow \frac{du}{dx} = ((x\cos x)^x)[1 - x\tan x + (\log x(x\cos x)] \qquad \ldots\ldots. \quad (2)$$

Also we know, $v = (x \sin x)^{\frac{1}{x}}$

$$\Rightarrow \log v = \log(x \sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log v = \frac{1}{x} \log(x \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} (\log x + \log \sin x)$$

$$\Rightarrow \log v = \frac{1}{x} \log x + \frac{1}{x} \log \sin x$$

Differentiating both sides of equation w.r.t. x gives

$$\Rightarrow \frac{1}{v} \frac{d}{dx} = \frac{d}{dx}\left(\frac{1}{x} \log x\right) + \frac{d}{dx}\left[\frac{1}{x} \log(\sin x)\right]$$

$$\Rightarrow \frac{1}{v} \frac{d}{dx} = \left[\frac{1}{x} \log x \times \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x} \times \frac{d}{dx} (\log x)\right] + \left[\log(\sin x) \times \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x} \times \frac{d}{dx} \{(\log \sin x)\}\right]$$

$$\Rightarrow \frac{1}{v}\frac{d}{dx} = \left[\frac{1}{x}\log x \times \left(-\frac{1}{x^2}\right) + \frac{1}{x} \times \frac{1}{x}\right]$$
$$+ \left[\log(\sin x) \times \left(-\frac{1}{x^2}\right)\right.$$
$$\left. + \frac{1}{x} \times \frac{1}{\sin x} \times \frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(1 - \log x) + \left[\frac{1 - \log x}{x^2} + \frac{1}{x \sin x} \times \cos x\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1}{x^2}(x \sin x)^{\frac{1}{x}} + \left[\frac{1 - \log x}{x^2} + \frac{-\log(\sin x) + x \cot x}{x^2}\right]$$

$$\frac{dv}{dx} = (x \sin x)^{\frac{1}{x}}\left[\frac{1 - \log x - \log(\sin x) + x \cot x}{x^2}\right]$$

$$\Rightarrow \frac{dv}{dx} = (x \sin x)^{\frac{1}{x}}\left[\frac{1 - \log(x \sin x) + x \cot x}{x^2}\right] \qquad \ldots..(3)$$

Hence, from the equations (1), (2) and (3), gives

$$\Rightarrow \frac{dv}{dx} = (x \cos x)^2[1 - x \tan x + log(x \cos x)]$$
$$+ (x \sin x)^{\frac{1}{x}}\left[\frac{1 - \log(x \sin x) + x \cot x}{x^2}\right]$$

12.

Find $\frac{dy}{dx}$ of the function $x^y + y^x = 1$

Ans - Given function is $x^y + y^x = 1$.

Let $x^y = u$ and $y^x = v$

$\Rightarrow u + v = 1$

Differentiating both sides of equation w.r.t. x gives

$$\frac{du}{dx} + \frac{dv}{dy} = 0$$

We know, $u = x^y$ ... (1)

$\Rightarrow \log u = \log(x^y)$

$\Rightarrow \log u = y \log x$

Differentiating both sides of equation w.r.t. x gives

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y \times \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{du}{dx} = x^y\left[\log x \frac{dy}{dx} + \frac{y}{x}\right] \quad \ldots\ldots\ldots \quad (2)$$

Also we know, $v = y^x$

Taking logarithm on both sides of equation gives

$\Rightarrow \log v = \log(y^3)$

$\Rightarrow \log v = x \log y$

Differentiating both sides of equation w.r.t. x gives

$$\frac{1}{u} \times \frac{dv}{dx} = \log y \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\log y)$$

$$\Rightarrow \frac{dv}{dx} = v\left(\log y \times 1 + x \times \frac{1}{y} \times \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad \ldots\ldots \quad (3)$$

Hence, from equations (1), (2) and (3), we get

$$x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) + y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) = 0$$

$$\Rightarrow (x^2 + \log x + xy^{y-1})\frac{dv}{dx} = -(yx^{y-1} + y^x \log y)$$

$$\therefore \frac{dy}{dx} = \frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

13.

Find $\frac{dy}{dx}$ of the function $y^x = x^y$.

Ans - Given equation is $y^x = x^y$.

Taking logarithm on both sides of the equation give

$x \log y = y \log x$.

Differentiating both sides of equation w.r.t. x gives.

$$\log y \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\log y) = \log x \times \frac{d}{dx}(y) + y \times \frac{d}{dx}(\log x)$$

$$\Rightarrow \log y \times 1 + x \times \frac{1}{y} \times \frac{dy}{dx} = \log x \times \frac{dy}{dx} + y \times \frac{1}{x}$$

$$\Rightarrow \log y + \frac{x}{y}\frac{dy}{dx} = \frac{y}{x} - \log y$$

$$\Rightarrow \left(\frac{x - y\log x}{y}\right)\frac{dy}{dx} = \frac{y - x\log y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}\left(\frac{y - x\log y}{x - y\log x}\right)$$

14.

Find $\frac{dy}{dx}$ of the function $(\cos x)^y = (\cos y)^x$.

Ans - Given equation is $(\cos x)^y = (\cos y)^x$.

Taking logarithm on both sides of the equation give

$y \log \cos x = x \log \cos y$

Differentiating both sides of equation w.r.t. x gives.

$$\log \cos x \times \frac{dy}{dx} + y \times \frac{d}{dx}(\log \cos x)$$
$$= \log \cos y \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\log \cos y)$$

$$\Rightarrow \log \cos x \frac{dy}{dx} + \frac{y}{\cos x} \times (-\sin x)$$
$$= \log \cos y + \frac{x}{\cos y}(-\sin y) \times \frac{dy}{dx}$$

$$\Rightarrow \log \cos x \frac{dy}{dx} - y \tan x = \log \cos y - x \tan y \frac{dy}{dx}$$

$$\Rightarrow (\log \cos x + x \tan y) \frac{dy}{dx} = y \tan x + \log \cos y$$

$\therefore \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$

15.

Find $\frac{dy}{dx}$ of the function $xy = e^{(x-y)}$.

Ans - Given equation is $xy = e^{(x-y)}$

Taking logarithm on both sides of the equation give

$$\log(xy) = \log(e^{(x-y)})$$

$$\Rightarrow \log x = \log y = (x - y)\log e$$

$$\Rightarrow \log x + \log y = (x - y) \times 1$$

$$\Rightarrow \log x + \log y = x - y$$

Differentiating both sides of equation w.r.t. x gives.

$$\frac{d}{dx}(\log x) + \frac{d}{dx}(\log y) = \frac{d}{dx}(x) - \frac{dy}{dx}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\Rightarrow \left(1 + \frac{1}{y}\right)\frac{dy}{dx} = \frac{x-1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(x+1)}$$

16.

Find the derivative of the function given by $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ and hence find $f'(1)$.

Ans - Given function is $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$.

By taking logarithm on both sides of the equation give.

$$\log f(x) = \log(1 + x)(1 + x^2)(1 + x^4)(1 + x^8).$$

$$\log f(x) = \log(1 + x) + \log(1 + x^2) + \log(1 + x^4) + \log(1 + x^8)$$

Differentiating both sides of equation w.r.t. x gives

$$\frac{1}{f(x)} \times \frac{d}{dx}[f(x)] = \frac{d}{dx}\log(1 + x) + \frac{d}{dx}\log(1 + x^2) + \frac{d}{dx}\log(1 + x^4) + \log(1 + x^8)$$

$$\Rightarrow \frac{1}{f(x)} \times f'(x) = \frac{1}{1 + x} \times \frac{1}{dx}(1 + x) + \frac{1}{1 + x^2} \times \frac{d}{dx}\log(1 + x^2) + \frac{1}{1 + x^4} \times \frac{d}{dx}\log(1 + x^4) + \frac{1}{1 + x^8} \times \frac{d}{dx}\log(1 + x^8)$$

$$\Rightarrow f'(x) = f(x)\left[\frac{1}{1 + x} + \frac{1}{1 + x^2} \times 2x + \frac{1}{1 + x^4} \times 4x^3 + \times \frac{1}{1 + x^8} \times 8x^7\right]$$

Therefore,

$$f'(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\left[\frac{1}{1 + x} + \frac{2x}{1 + x^2} + \frac{4x^3}{1 + x^4} + \frac{8x^7}{1 + x^8}\right]$$

So,

$$f'(1) = (1+1)(1+1^2)(1+1^4)(1 + 1^8)\left[\frac{1}{1+1} + \frac{2\times 1}{1+1^2} + \frac{4\times 1^3}{1+1^4} + \frac{8\times 1^7}{1+1^8}\right]$$

$$= 2\times 2\times 2\times 2\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$$

$$= 16 \times \left(\frac{1+2+4+8}{2}\right)$$

$$= 16 \times \frac{15}{2} = 120$$

$$\therefore \{f\}'(1) = 120$$

17.

Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways as mentioned below:
(i) by using product rule
(ii) by expanding the product to obtain a single polynomial.
(iii) by logarithmic differentiation.
Do they all give the same answer?

Ans – (i) Given function is $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$.

Let $u = (x^2 - 5x + 8)$ and $v = (x^3 + 7x + 9)$

$\Rightarrow y = uv$.

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dv}.v + u.\frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 - 5x + 8).(x^3 + 7x + 9) + (x^2 - 5x + 8).\frac{d}{dx}(x^3 + 7x + 9)$$

$$\Rightarrow \frac{dy}{dx} = (2x - 5)(x^3 + 7x + 9).(x^2 - 5x + 8)(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = 2x(x^3 + 7x + 9) - 5(x^2 - 5x + 8) + x^2(3x^2 + 7) - 5x(3x^2 + 7) - 8(3x^2 + 7)$$

$$\Rightarrow \frac{dy}{dx} = (2x^4 + 14x^2 + 18x) - 5x^3 - 35x - 45 + (3x^4 + 7x^2) - 15x^3 - 35x + 24x^2 + 56$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 + 52x + 11$$

(ii) Given function is $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$.

Calculating the product will give,

$$y = x^2(x^3 + 7x + 9) - 5x^4(x^3 + 7x + 9) + 8(x^3 + 7x + 9)$$

$$\Rightarrow y = x^5 + 7x^3 + 9x^2 - 5x^3 - 26x^2 + 11x + 72$$

Differentiating both sides of equation w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}(x^5 + 7x^3 + 9x^3 - 5x^3 - 26x^3 + 11x + 72)$$

$$= \frac{d}{dx}(x^5) - 5\frac{d}{dx}(x^4) + 15\frac{d}{dx}(x^3) - 26\frac{d}{dx}(x^3) + 111\frac{d}{dx}(x) + \frac{d}{dx}(72)$$

$$= 5x^4 - 5 \times 4x^3 + 15 \times 3x^2 - 26 \times 2x + 11 \times 1 + 0$$

$$\therefore \frac{dy}{dx} = 5x^4 - 20x^3 + 45x^2 - 52x + 11$$

(iii) Given function is $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$.

Taking logarithm both sides of the equation will give,

$$\log y = \log(x^2 - 5x + 8) + \log(x^3 + 7x + 9)$$

Difference both sides of equation w.r.t. x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\log(x^2 - 5x + 8) + \frac{d}{dx}\log(x^3 + 7x + 9)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 - 5x + 8}.\frac{d}{dx}(x^2 - 5x + 8) + \frac{1}{x^3 + 7x + 9}.\frac{d}{dx}(x^3 + 7x + 9)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x^2 - 5x + 8} \times (2x - 5) + \frac{1}{x^3 + 7x + 9} \times (3x^2 + 7)\right]$$

$$\Rightarrow \frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9)\left[\frac{2x - 5}{x^3 - 5x + 8} + \frac{3x^2 + 7}{x^3 + 7x + 9}\right]$$

$$\Rightarrow \frac{dy}{dx} = (x^2 - 5x + 8)(x^3 + 7x + 9)$$

$$\left[\frac{(2x - 5)(x^3 + 7x + 9) + (3x^2 + 7)(x^3 - 5x + 8)}{(x^3 - 5x + 8)(x^3 + 7x + 9)}\right]$$

$$\Rightarrow \frac{dy}{dx} = 2x(x^3 + 7x + 9x^2) - 5(x^3 + 7x + 9)$$
$$+ 3x^2(x^2 - 5x + 8) + 7(x^3 + 7x + 9)$$

$$\Rightarrow \frac{dy}{dx} = (2x^4 + 14x^4 + 18x) + (5x^3 - 35x + 45)$$
$$+ (3x^4 - 15x^3 + 24x^2) + (7x^2 + 35x + 56)$$

$$\therefore \frac{dy}{dx} = 5x^2 - 20x^3 + 45x^2 - 52x + 11$$

Hence, comparing the above results, it is concluded that the derivative $\frac{dy}{dx}$ are the same for all methods.

18.

If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.v.w) = \frac{d}{dx}v.w + u\frac{du}{dx}.w + u.v\frac{dw}{dx}$$

in two ways - first by repeated application of product rule, second by logarithmic differentiation

Ans - Let $y = u.v.w = u.(v.w)$.

Applying product rule of derivatives, we get

$$\frac{dy}{dx} = \frac{du}{dx}.(v.w) + u.\frac{d}{dx}(v.w)$$

Using the product rule again we get

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}v.w + u.\left[\frac{dv}{dx}.w + u.v\frac{d}{dx}\right]$$

Now, take logarithm of both sides of function $y = u.v.w$.

$$\Rightarrow \log y = \log u + \log v + \log w$$

Differentiating both sides of equation w.r.t. x gives.

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\log u) + u.\frac{dv}{dx}(\log v) + \frac{d}{dx}(\log w)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = u.v.w\left(\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx}v.w + u\frac{du}{dx}.w + u.v\frac{dw}{dx}$$

Exercise 5.6

1.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$$\mathbf{x = 2at^2, y = at^4}$$

Ans - Given equations are

$x = 2at^2$ ..... (1)

$y = at^4$ .... (2)

Differentiating both sides of equation (1) w.r.t. x gives

$$\frac{dx}{dy} = \frac{d}{dt}(2at^2)$$

$$\Rightarrow \frac{dx}{dt} = 2a\frac{d}{dt}(t^2) = 2a \times 2t = 4at \dots\dots\dots\dots (3)$$

Differentiating both sides of equation (2) w.r.t. t gives

$$\frac{dy}{dt} = \frac{d}{dt}(2at^4) = a + 4 + \frac{d}{dt}(t^4)$$

$$\Rightarrow \frac{dy}{dt} = a \times 4 \times t^3 = 4at^3 \quad \dots\dots\dots \ (4)$$

Now, dividing the equations (4) by (3) gives

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{4at^3}{4at} = t^2$$

$$\therefore \frac{dy}{dx} = t^2$$

2.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$\mathbf{x = a\cos\theta\,, y = b\cos\theta}$

Ans - Given equations are

$x = a\cos\theta \qquad \ldots\ldots \quad (1)$

$y = b\cos\theta \qquad \ldots\ldots \quad (2)$

Differentiating both sides of equation (1) w.r.t. θ gives

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a(-\sin\theta) = -a\sin\theta \qquad \ldots\ldots \quad (3)$$

Differentiating both sides of equation (2) w.r.t. θ gives

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\cos\theta)$$

$$\Rightarrow \frac{dy}{d\theta} = b(-\sin\theta) = -b\sin\theta \qquad \ldots\ldots \quad (4)$$

Now, dividing equation (4) by (3) gives

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{dy}{dx} = \frac{-b\sin\theta}{-a\sin\theta} = \frac{b}{a}$$

$$\therefore \frac{dy}{dx} = \frac{b}{a}$$

3.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$\mathbf{x = \sin\ t, y = \cos\ 2t}$

Ans - Given equations are

$x = \sin\ t$ ...... (1)

$y = \cos\ 2t$ ...... (2)

Differentiating both sides of equation (1) w.r.t. t gives

$$\frac{dx}{dt} = \frac{d}{dt}(\sin\ \theta) = \cos\ \theta \ldots\ldots (3)$$

Differentiating both sides of equation (2) w.r.t. t gives

$$\frac{dy}{dt} = \frac{d}{dt}(\cos 2\theta) = \sin 2\theta \times \frac{d}{dt}(2t) = -2\sin 2t \quad \ldots\ldots (4)$$

Now, by dividing the equation (4) by (3) gives

$$\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx} = \frac{-2\sin\ 2t}{\cos\ t} = \frac{-2 \times 2\sin\ t\cos\ t}{\cos\ t} = -4\sin\ t$$

$$\therefore \frac{dy}{dx} = -4\sin\ t$$

4.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$$\mathbf{x = 4t, y = \frac{4}{t}}$$

Ans - Given equations are

$x = 4t \qquad ......(1)$

$y = \frac{4}{t} \qquad ......(2)$

Differentiating both sides of equation (1) w.r.t. t gives

$\frac{dy}{dx} = \frac{d}{dt}(4t) = 4 \qquad ......(3)$

Differentiating both sides of equation (2) w.r.t. t gives

$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{4}{t}\right) = 4 \times \frac{d}{dt}\frac{1}{t} = 4 \times \frac{-1}{t^2} = \frac{-4}{t^2} \qquad ......(4)$

Now, dividing the equation (4) by (3) gives

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dz}{dt}\right)} = \frac{\left(\frac{-4}{t^2}\right)}{4} = \frac{-1}{t^2}$$

$$\frac{dy}{dx} = \frac{-1}{t^2}$$

5.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$\mathbf{x = \cos\ \theta - \cos\ 2\theta, y = \sin\ \theta - \sin\ 2\theta}$

Ans - Given equations are

$x = \cos\theta - \cos 2\theta \qquad \ldots\ldots(1)$

$y = \sin\ \theta - \sin\ 2\theta \qquad \ldots\ldots(2)$

Differentiating both sides of the equation (1) w.r.t. θ gives

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\cos\ \theta - \cos\ 2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{dt}(\cos\ \theta) - \frac{d}{d\theta}(\cos\ 2\theta)$$

$$= -\sin\ \theta(-2\sin\ 2\theta) = 2\sin\ 2\theta - \sin\ \theta \quad \ldots\ldots(3)$$

Differentiating both sides of equation (2) w.r.t. θ gives

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\theta - \sin 2\theta)$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{d}{d\theta}(\sin\ \theta) - \frac{d}{d\theta}(\sin\ 2\theta)$$

$$= \cos\ \theta - 2\cos\ \theta \quad \ldots\ldots(4)$$

Now, dividing the equation (4) by (3) gives

$$\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{dy}{dx} = \frac{\cos\ \theta - 2\cos\ \theta}{2\sin\ 2\theta - \sin\ \theta}$$

$$\therefore\ \frac{dy}{dx} = \frac{\cos\ \theta - 2\cos\ \theta}{2\sin\ 2\theta - \sin\ \theta}$$

6.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$\mathbf{x = a(\theta - \sin\ \theta), y = a(1 + \cos\ \theta)}$

Ans - Given equations are

$x = a(\theta - \sin\ \theta)$ ...... (1)

$y = a(1 + \cos\ \theta)$ ...... (2)

Differentiating both sides of equation (1) w.r.t. θ gives

$$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}(\theta) - \frac{d}{d\theta}(\sin\ \theta)\right] = a(1 - \cos\ \theta) \quad ...... (3)$$

Also, differentiating both sides of the equation (2) w.r.t. θ gives

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(1) - \frac{d}{d\theta}(\cos\theta)\right]$$

$$\Rightarrow \frac{dy}{d\theta} = a[0 + (-\sin\ \theta)] = -a\sin\ \theta \quad ...... (4)$$

Now, by dividing the equation (4) by (3) gives

$$\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{dy}{dx} = \frac{-a\sin\theta}{a(1 - \cos\theta)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cos\frac{\theta}{2}$$

$$\therefore \frac{dy}{dx} = -\cos\frac{\theta}{2}$$

7.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$$-\frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

Ans - Given equations are,

$$x=-\frac{\sin^3 t}{\sqrt{\cos 2t}} \quad \dots\dots(1)$$

$$y=-\frac{\cos^3 t}{\sqrt{\cos 2t}} \quad \dots\dots(2)$$

Differentiating both sides of equation (1) w.r.t. t gives

$$\frac{dx}{dt}=\frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}-\frac{d}{dt}(\sin^3 t)-\sin^3 t\times\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$=\frac{\sqrt{\cos 2t}\times 3\sin^2 t\times\frac{d}{dt}(\sin t)-\sin^3 t\times\frac{1}{2\sqrt{\cos 2t}}\times\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t}\times\sin^2 t\cos t-\frac{\sin^3 t}{2\sqrt{\cos 2t}}\times(-2\sin 2t)}{\cos 2t\sqrt{\cos 2t}}$$

$$\Rightarrow\frac{dx}{dt}=\frac{3\cos 2\sin^2 t\cos t+\sin^2 t\sin 2t}{\cos 2t\sqrt{\cos 2t}} \quad \dots\dots (3)$$

Differentiating both sides of equation (2) w.r.t. t gives

$$\frac{dy}{dt}=\frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$=\frac{\sqrt{\cos 2t}\times\frac{d}{dt}(\cos^3 t)-\cos^3 t\times\frac{d}{dt}(\sqrt{\cos 2t})}{\cos 2t}$$

$$=\frac{3\sqrt{\cos 2t\cos^2 t(-\sin t)-\cos^3 t\times\frac{1}{2(\sqrt{\cos 2t})}}\times\frac{d}{dt}(\cos 2t)}{\cos 2t}$$

$$\Rightarrow\frac{dy}{dt}=\frac{-3\cos 2t\times\cos^2 t\times\sin t+\cos^3 t\sin 2t}{\cos 2t\times\sqrt{\cos 2t}} \quad \dots\dots (4)$$

Thus, dividing the equation (4) by the equation (3) gives

$$\frac{\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx} = \frac{-3\cos\ 2t \times \cos^2\ t \times \sin\ t + \cos^3\ t\sin\ 2t}{3\cos\ 2t\cos\ t\sin^2\ t + \sin^3\ t\sin\ 2t}$$

$$= \frac{\sin\ t\cos\ t[-3\cos\ 2t \times \cos\ t + 2\cos^3\ t]}{\sin\ t\cos\ t[3\cos\ 2t\sin\ t + 2\sin^3\ t]}$$

$$= \frac{[-3(2\cos^2\ t - 1)\cos\ t + 2\cos^3\ t]}{[3(1 - 2\sin^3\ t)\sin\ t + 2\sin^3\ t]} \begin{bmatrix}\cos\ 2t = (2\cos^2\ t - 1) \\ \cos\ 2t = (1 - 2\sin^2\ t)\end{bmatrix}$$

$$= \frac{-4\cos^3\ t + 3\cos\ t}{3\sin\ t - 4\sin^3\ t}$$

$$= \frac{-\cos\ 3t}{\sin\ 3t}$$

$$\Rightarrow \frac{dy}{dx} = -\cot\ 3t$$

8.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$$\mathbf{x = a\left(\cos\ t + \log\ \tan\ \frac{t}{2}\right), y = a\sin\ t}$$

Ans - Given equations are

$$x = a\left(\cos\ t + \log\ \tan\ \frac{t}{2}\right) \quad \ldots\ldots(1)$$

$$y = a\sin\ t \quad \ldots\ldots(2)$$

Differentiating both sides of equation (1) w.r.t. t gives

$$\frac{dx}{dt} = a \times \left[\frac{d}{d\theta}(\cos\ t) + \frac{d}{d\theta}\left(\log\ \tan\ \frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}} \times \frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$= a\left[-\sin\ t + \cot\ \frac{t}{2} \times \sec^2\ \frac{t}{2} \times \frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin\ t + \frac{\cos\ \frac{t}{2}}{\sin\ \frac{t}{2}} \times \frac{1}{\cos^2\ \frac{t}{2}} \times \frac{1}{2}\right)$$

$$= a\left(-\sin\ t + \frac{1}{2\sin\ \frac{t}{2}\cos\ \frac{t}{2}}\right)$$

$$= a\left(-\sin\ t + \frac{1}{\sin\ t}\right)$$

$$= a\left(\frac{-\sin^2\ t + 1}{\sin\ t}\right)$$

$$\Rightarrow \frac{dx}{dt} = a\frac{\cos^2\ t}{\sin\ t} \quad \ldots\ldots(3)$$

Differentiating both sides of equation (2) w.r.t. t gives

$$\frac{dy}{dt} = a\frac{d}{dt}(\sin\ t) = a\cos\ t \quad \ldots\ldots(4)$$

Now, dividing the equation (4) by the equation (3) gives

$$\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx} = \frac{a\cos\ t}{\left(a\frac{\cos^2\ t}{\sin\ t}\right)} = \frac{\sin\ t}{\cos\ t} = \tan t$$

$$\therefore\ \frac{dy}{dx} = \tan\ t$$

9.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$\mathbf{x = a\sec\theta, y = b\tan\ \theta}$

Ans - Given equations are

$x = a\sec\theta \quad \ldots\ldots\ (1)$

$y = \tan\ \theta \quad \ldots\ldots\ (2)$

Differentiating both sides of equation (1) w.r.t. $\boldsymbol{\theta}$ gives

$$\frac{dx}{d\theta} = a \times \frac{d}{d\theta}(\sec\ \theta) = a\sec\theta\tan\ \theta \quad \ldots\ldots\ (3)$$

Differentiating both sides of equation (2) w.r.t. $\boldsymbol{\theta}$ gives

$$\frac{dy}{d\theta} = b \times \frac{d}{d\theta}(\tan\ \theta) = b\sec^2\theta \quad \ldots\ldots\ (4)$$

Thus, dividing the equation (4) by the equation (3) gives

$$\frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\ \theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{b}{a}\sec\ \theta\tan\ \theta = -\frac{b\cos\ \theta}{a\cos\ \theta\sin\ \theta} = \frac{b}{a}\times\frac{1}{\sin\ \theta}$$
$$= \frac{b}{a}\text{cosec}\theta$$

$$\frac{dy}{dx} = \frac{b}{a}\text{cosec}\theta$$

10.

If x and y are connected parametrically by the equation given without eliminating the parameter, Find $\frac{dy}{dx}$.

$\mathbf{x = a(\cos\ \theta + \theta\sin\ \theta)}$, $\mathbf{y = a(\sin\ \theta - \theta\cos\ \theta)}$

Ans - Given equations are

$x = a(\cos\ \theta + \theta\sin\ \theta)$ ... ... (1)

$y = a(\sin\ \theta - \theta\cos\ \theta)$ ...... (2)

Differentiating both sides of equation (1) w.r.t. θ gives

$$\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\ \theta + \frac{d}{d\theta}(\theta\sin\ \theta)\right]$$

$$\frac{dx}{d\theta} = a\left[-\sin\ \theta + \theta\frac{d}{d\theta}(\sin\ \theta) + \sin\ \theta\frac{d}{d\theta}(\theta)\right]$$

$$= a[-\sin\ \theta + \theta\cos\ \theta + \sin\ \theta]$$

$$\Rightarrow \frac{dx}{d\theta} = a\theta\cos\ \theta \quad ......(3)$$

Differentiating both sides of equation (2) w.r.t. θ gives

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\ \theta) - \frac{d}{d\theta}(\theta\cos\ \theta)\right]$$

$$= a\left[\cos\ \theta - \left\{\theta\frac{d}{d\theta}(\cos\ \theta) + \cos\ \theta \times \frac{d}{d\theta}(\theta)\right\}\right]$$

$$\Rightarrow \frac{dy}{d\theta} = a[\cos\ \theta + \theta\sin\ \theta - \cos\ \theta]$$

$$\Rightarrow \frac{dy}{d\theta} = a\theta\sin\ \theta\ \ \ldots\ldots(4)$$

Now, dividing equation (4) by equation (3) gives

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dz}{d\theta}\right)} = \frac{a\theta\sin\ \theta}{a\theta\sin\ \theta} = \tan\ \theta$$

$$\frac{dy}{dx} = \tan\ \theta$$

11.

If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$. Show that $\frac{dy}{dx} = -\frac{y}{x}$

Ans - Given parametric equations are

$x = \sqrt{a^{\sin^{-1} t}}$

$y = \sqrt{a^{\cos^{-1} t}}$

Now, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

$\Rightarrow x = \left(a^{\sin^{-1} t}\right)$ and $y = \left(a^{\cos^{-1} t}\right)^{\frac{1}{2}}$

$\Rightarrow x = a^{\frac{1}{2}\sin^{-1}t}$ and $y = a^{\frac{1}{2}\cos^{-1} t}$

First consider $x = a^{\frac{1}{2}\sin^{-1} t}$

Take logarithm on both sides of equation, we get,

$\log x = \frac{1}{2}\sin^{-1} t \log a$

Differentiating both sides of equation w.r.t. t gives

$$\frac{1}{x} \times \frac{dx}{dt} = \frac{1}{2}\log a \times \frac{d}{dt}(\sin^{-1} t)$$

$$\Rightarrow \frac{dx}{dt} = \frac{x}{2}\log a \times \frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{x \log a}{2\sqrt{1-t^2}} \qquad \ldots\ldots. (1)$$

Now, consider equation $y = a^{\frac{1}{2}\cos^{-1} t}$

Take logarithm on both sides of equation, we get,

$\log\ y = \frac{1}{2}\cos^{-1} t \log a$

Differentiating both sides of the equation w.r.t. t gives

$$\frac{1}{y} \times \frac{dx}{dt} = \frac{1}{2}\log\ a \times \frac{d}{dt}(\cos^{-1}\ t)$$

$$\frac{1}{y} \times \frac{dx}{dt} = \frac{1}{2}\log\ a \times \frac{d}{dt}(\cos^{-1}\ t)$$

$$\Rightarrow \frac{dx}{dt} = \frac{y\log\ a}{2} \times \left(\frac{1}{\sqrt{1-t^2}}\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{-y\log a}{2\sqrt{1-t^2}} \qquad \ldots\ldots.(2)$$

Now, dividing equation (2) by equation (1) gives

$$\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx} = \frac{\left(\frac{-y\log\ a}{2\sqrt{1-t^2}}\right)}{\left(\frac{x\log}{2\sqrt{1-t^2}}\right)} = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}$$

Exercise 5.7

1.

Find the second order derivatives of the function given:
$\mathbf{y = x^2 + 3x + 2.}$

Ans - Given function is $y = x^2 + 3x + 2$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0$$

$$\Rightarrow \frac{dy}{dx} = 2x + 3$$

Differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

$$\therefore \frac{d^2y}{dx^2} = 2$$

2.

Find the second order derivatives of the function given:
$\mathbf{y = x^{20}}$

Ans - Given function is $y = x^{20}$

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}(x^{20}) = 20x^{19}$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 20\frac{d}{dx}(x^{19}) = 20(19)x^{18} = 380x^{18}$$

$$\therefore \frac{d^2y}{dx^2} = 380x^{18}$$

3.

Find the second order derivatives of the function given $y = x \cdot \cos\ x$.

Ans - Given function is $y = x \cdot \cos\ x$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \cos\ x) = \cos\ x \cdot \frac{d}{dx}(x) + x\frac{d}{dx}(\cos\ x)$$
$$= \cos\ x \cdot 1 + x(-\sin\ x)$$

$$\Rightarrow \frac{dy}{dx} = \cos\ x - x\sin\ x$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos\ x - x\sin\ x) = \frac{d}{dx}(\cos\ x) - \frac{d}{dx}(x\sin\ x)$$

$$= -\sin\ x - \left[\sin\ x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin\ x)\right]$$

$$= -\sin\ x - (\sin\ x + x\cos\ x)$$

$$\therefore \frac{d^2y}{dx^2} = -(x\cos\ x + 2\sin\ x)$$

4.

Find the second order derivatives of the function given $y = \log\ x$.

Ans - Given function is $y = \log x$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

5.

Find the second order derivatives of the function given $y = x^3 \log x$.

Ans - Given function is $y = x^3 \log x$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}[x^3 \log x] = \log x \cdot \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}(\log x)$$
$$= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2$$

$$\Rightarrow \frac{dy}{dx} = x^2(1 + 3\log x)$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(x^2(1 + 3\log x))$$

$$= (1 + 3\log x) \cdot \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(1 + 3\log x)$$

$$= (1 + 3\log x) \cdot 2x + x^3 \cdot \frac{3}{x}$$

$$= 2x + 6\log x + 3x$$

$$= 5x + 6x \log x$$

$$\therefore \frac{d^2y}{dx^{2^2}} = x(5 + 6\log x)$$

6.

Find the second order derivatives of the function given
$\mathbf{y = e^x \sin 5x}$

Ans - Given function is $y = e^x \sin 5x$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}[e^x \sin 5x] = \sin x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\sin 5x)$$

$$\Rightarrow \frac{dy}{dx} = \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx}(5x)$$

$$\Rightarrow \frac{dy}{dx} = e^x(\sin 5x + 5\cos 5x)$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[e^x(\sin 5x + 5\cos 5x)]$$

$$= (\sin 5x + 5\cos 5x) \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(\sin 5x + 5\cos 5x)$$

$$= (\sin 5x + 5\cos 5x)(e^x) + e^x\left[\cos 5x \cdot \frac{d}{dx}(5x) + 5(-\sin 5x) \cdot \frac{d}{dx}(5x)\right]$$

$$= e^x(\sin 5x + 5\cos 5x) + e^x(5\cos 5x - 25\sin 5x)$$

$$= e^x(10\cos 5x - 24\sin 5x)$$

$$\therefore \frac{d^2y}{dx^2} = 2e^x(5\cos 5x - 12\sin 5x)$$

7.

Find the second order derivatives of the function given $y = e^{6x}\cos 3x$.

Ans - Given function is $y = e^{6x}\cos 3x$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}(e^{6x}\cos 3x)$$
$$= \cos 3x \times \frac{d}{dx}(e^{6x}) + e^{6x} \times \frac{d}{dx}(\cos 3x)$$

$$\Rightarrow \frac{dy}{dx} = \cos 3x \times e^{6x} \times \frac{d}{dx}(6x) + e^{6x} \times (-\sin 3x) \times \frac{d}{dx}(3x)$$

$$\Rightarrow \frac{dy}{dx} = 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \quad \ldots\ldots (1)$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(6e^{6x}\cos 3x - 3e^{6x}\sin 3x)$$
$$= 6 \times \frac{d}{dx}(e^{6x}\cos 3x) - 3 \times \frac{d}{dx}(e^{6x}\sin 3x)$$

$$= 6 \times [6e^{6x}\cos 3x - 3e^{6x}\sin 3x]$$
$$-3 \times \left[\sin 3x \times \frac{d}{dx}(e^{6x}) + e^{6x} \times \frac{d}{dx}(\sin 3x)\right] \text{using}(1)$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x$$
$$- 3[\sin 3x \times e^{6x} \times 6 + e^{6x} \times \cos 3x - 3]$$

$$= 36e^{6x}\cos 3x - 18e^{6x}\sin 3x - 18e^{6x}\sin 3x - 9e^{6x}\cos 3x$$

$$\therefore \frac{d^2y}{dx^2} = 9e^{6x}(3\cos 3x - 4\sin 3x)$$

8.

Find the second order derivatives of the function given $y = \tan^{-1}x$.

Ans - Given function is $y = \tan^{-1} x$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}\tan^{-1} x = \frac{1}{1 - x^2}$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{d^2x} = \frac{d}{dx}\left(\frac{1}{1 + x^2}\right) = \frac{d}{dx}(1 + x^2)^{-1}$$

$$= (-1) \times (1 + x^2)^{-2} \times \frac{d}{dx}(1 + x^2)$$

$$= -\frac{1}{(1 + x^2)^2} \times 2x$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-2x}{(1 + x^2)^2}$$

9.

Find the second order derivatives of the function given $y = \log(\log x)$.

Ans - Given function is $y = \log(\log x)$

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\log x)] = \frac{1}{\log x} \times \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{dy}{dx} = (x\log x)^{-1}$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[(x\log x)^{-1}] = (-1) \times (x\log x)^{-2} \frac{d}{dx}(x\log x)$$

$$= \frac{-1}{(x\log x)^2} \times \left[\log x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\log x)\right]$$

$$= \frac{-1}{(x\log x)^2} \times \left[\log x \times 1 + x \times \frac{1}{x}\right]$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-(1 + \log x)}{(x\log x)^2}$$

10.

Find the second order derivatives of the function given $y = \sin(\log x)$.

Ans - Given function is $y = \sin(\log x)$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)] = \cos(\log x) \times \frac{d}{dx}(\log x)$$
$$= \frac{\cos(\log x)}{x}$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{d^2} = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$
$$= \frac{x[\cos(\log x)] - \cos(\log x) \times \frac{d}{dx}(x)}{x^2}$$
$$= \frac{x\left[-\sin(\log x) \times \frac{d}{dx}(\log x)\right] - \cos(\log x) \times 1}{x^2}$$
$$= \frac{-x\sin(\log x) \times \frac{1}{x} - \cos(\log x)}{x^2}$$
$$\therefore \frac{d^2y}{dx^2} = \frac{[-\sin(\log x) - \cos(\log x)]}{x^2}$$

11.

If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$.

Ans - Given equation is $y = 5\cos\ x - 3\sin\ x$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}(5\cos\ x) - \frac{d}{dx}(3\sin\ x) = 5\frac{d}{dx}(\cos\ x) - 3\frac{d}{dx}(\sin\ x)$$
$$= 5(-\sin\ x) - 3\cos\ x$$

$$\Rightarrow \frac{dy}{dx} = -(5\sin\ x + 3\cos\ x)$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[-(5\sin\ x + 3\cos\ x)]$$

$$= -\left[5 \times \frac{d}{dx}(\sin\ x) + 3 \times \frac{d}{dx}(\cos\ x)\right]$$

$$= [5\cos\ x + 3(-\sin\ x)]$$

$$= -y$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

12.

If $y = \cos^{-1}\ x$, Find $\frac{d^2y}{dx^2}$ in the terms of y alone.

Ans - Given function is $y = \cos^{-1} x$.

Differentiating both sides w.r.t. X gives

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{\frac{-1}{2}}$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(1-x^2)^{\frac{-1}{2}}\right]$$

$$= \left(\frac{-1}{2}\right) \times (1-x^2)^{\frac{-3}{2}} \times \frac{d}{dx}(1-x^2)$$

$$= \frac{1}{\sqrt[2]{(1-x^2)^3}} \times (-2x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x}{\sqrt{(1-x^2)^3}} \qquad (1)$$

We know, $y = \cos^{-1} x \Rightarrow x = \cos y$. Substituting $x = \cos y$ into equation (1), gives

$$\frac{d^2x}{dy^2} = \frac{-\cos y}{\sqrt{(1-\cos^2 y)^3}}$$

$$= \frac{-\cos y}{\sin^3 y}$$

$$= \frac{-\cos y}{\sin y} \times \frac{1}{\sin^2 y}$$

$$\therefore \frac{d^2y}{dx^{2^2}} = -\cot y \times \operatorname{cosec}^2 y$$

13.

If $\mathbf{y = 3\cos(\log x) + 4\sin(\log x)}$, show that $\mathbf{x^2y_2 + xy_1 + y = 0}$.

Ans - Given equations, $y = 3\cos(\log x) + 4\sin(\log x) \ldots (1)$

$x^2y_2 + xy_1 + y = 0 \quad \ldots (2)$

Differentiating both sides of equation (1) w.r.t. x gives

$$y_1 = 3 \times \frac{d}{dx}[\cos(\log x)] + 4 \times \frac{d}{dx}[\sin(\log x)]$$

$$\Rightarrow y_1 = 3 \times \left[-\sin(\log x) \times \frac{d}{dx}(\log x)\right] + 4 \times \left[\cos(\log x) \times \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$$\Rightarrow y_1 = \frac{4\cos(\log x) - 3\sin(\log x)}{x}$$

Again, differentiating both sides w.r.t. x gives

$$y_2 = \frac{d}{dx}\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x}\right)$$

$$= \frac{x \cdot \frac{d}{dx}[4\{\cos(\log x)\}\{3\sin(\log x)\}] - \{4\cos(\log x) - 3\sin(\log x)\} \times 1}{x^2}$$

$$= \frac{x\left[-4\sin(\log x)\frac{d}{dx}(\log x) - 3\cos(\log x)\frac{d}{dx}(\log x)\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{x\left[-4\sin(\log x) \cdot \frac{1}{x} - 3\cos(\log x) \cdot \frac{1}{x}\right] - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$= \frac{-4\sin(\log x) - 3\cos(\log x) - 4\cos(\log x) + 3\sin(\log x)}{x^2}$$

$$\Rightarrow y_2 = \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

Now, substituting the derivatives y_1, y_2 and y into LHS of equation (2) gives,

$$x^2y_2 + xy_1 + y$$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x\left(\frac{4\cos(\log x) - 3\sin(\log x)}{x^2}\right) + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) + 4\cos(\log x) - 3\sin(\log x) + 4\sin(\log x)$$

$$= 0$$

Hence, it has been proved that $x^2y_2 + xy_1 + y = 0$.

14.

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.

Ans - Given equations are $y = Ae^{mx} + Be^{nx} \quad \ldots.(1)$

$$\frac{d^2y}{{dx^2}^2} - (m+n)\frac{dy}{dx} + mny = 0 \qquad \ldots..(2)$$

Differentiating both sides of equation (1) w.r.t. x gives

$$\frac{dy}{dx} = A \cdot \frac{d}{dx}(e^{mx}) + B \cdot \frac{d}{dx}(e^{mx})$$

$$= A \cdot e^{mx} \cdot \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx)$$

$$\Rightarrow \frac{dy}{dx} = Ame^{mx} + Bne^{nx}$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(Ame^{mx} + Bne^{nx}) = Am \cdot \frac{d}{dx}(e^{mx}) + Bn \cdot \frac{d}{dx}(e^{nx})$$

$$= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx}(nx)$$

$$\Rightarrow \frac{d^2y}{dx^2} = Am^2e^{mx} + Bn^2e^{nx}$$

Substituting the derivatives y_1, y_2 and y into LHS of the equation (2) gives

$$\frac{d^2y}{d^2} - (m+n)\frac{dy}{dx} + mny$$

$$= Am^2ex^{mx} + Bn^2e^{nx} - (m+n)\cdot(Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx})$$

$$= Am^2ex^{mx} + Bn^2e^{nx} - Amex^{\,mx} + Bmne^{nx} + Amne^{mx} + Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$$

$$= 0$$

Hence, it has been proved that $\frac{d^2y}{dx^{2^2}} - (m+n)\frac{dy}{dx} + mny = 0.$

15.

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$.

Ans - Given equation is $y = 500e^{7x} + 600e^{-7x}$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = 500 \times (e^{7x}) + 600 \times \frac{d}{dx}(-7x)$$

$$\Rightarrow \frac{dy}{dx} = 500 \times e^{7x} \times \frac{d}{dx}(7x) + 600 \times e^{-7x} \times \frac{d}{dx}(-7x)$$

$$\Rightarrow \frac{dy}{dx} = 3500e^{7x} - 4200e^{-7x}$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = 3500 \times \frac{d}{dx}(e^{7x}) - 4200 \times \frac{d}{dx}(e^{-7x})$$

$$= 3500 \times e^{7x} \times \frac{d}{dx}(7x) - 4200 \times e^{-7x} \times \frac{d}{dx}(-7x)$$

$$= 7 \times 3500 \times e^{7x} + 7 \times 4200 \times e^{-7x}$$

$$= 49 \times 500e^{7x} + 49 \times 600e^{-7x}$$

$$= 49(500e^{7x} + 600e^{-7x})$$

$$\Rightarrow \frac{d^2y}{dx^2} = 49y \ \ (\textit{using the given equation})$$

Hence, it has been proved that $\frac{d^2y}{dx^2} = 49y$.

16.

If $e^y(x + 1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Ans - Given equation is $e^y(x+1) = 1$.

$$\Rightarrow e^y = \frac{1}{x+1}$$

So, taking logarithm on both sides of equation gives

$$y = \log \frac{1}{(x+1)}$$

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = (x+1)\frac{d}{dx}\left(\frac{1}{x+1}\right) = (x+1) \times \frac{-1}{(x+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{x+1}$$

Again, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx} = \left(\frac{1}{x+1}\right) = -\left(\frac{-1}{(x+1)^2}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{-1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

Hence proved.

17.

If $y = (\tan^{-1} x)^2$, show that $(x^2+1)^2 y_2 + 2x(x^2+1)y_1 = 2$.

Ans - Given equation is $y = (\tan^{-1} x)^2$.

Differentiating both sides w.r.t. x gives

$$y_1 = 2\tan^{-1} x \frac{d}{dx}(\tan^{-1} x)$$

$$\Rightarrow y_1 = 2\tan^{-1} x \times \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y_1 = 2\tan^{-1} x$$

Again, differentiating both sides w.r.t. x gives

$$(1+x^2)y_2 + 2xy_1 = 2\left(\frac{1}{1+x^2}\right)$$

$$\Rightarrow (1+x^2)y_2 + 2x(1+x^2)y_1 = 2$$

Hence proved.

Miscellaneous Exercise

Differentiate w.r.t. x the function in Exercises 1 to 11

1.

$\mathbf{y = (3x^2 - 9x + 5)^9}$.

Ans - Given function is $y = (3x^2 - 9x + 5)^9$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}(3x^2 - 9x + 5)^9$$

$$= 9(3x^2 - 9x + 5)^8 \times \frac{d}{dx}(3x^2 - 9x + 5)$$

$$= 9(3x^2 - 9x + 5)^8 \times (6x - 9x)$$

$$= 9(3x^2 - 9x + 5)^8 \times 3(2x - 3)$$

$$= 27(3x^2 - 9x + 5)^8(2x - 3)$$

2.

$\mathbf{y = \sin^3 x + \cos^6 x.}$

Ans - Given function is $y = \sin^3 x + \cos^6 x$.

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx} = (\sin^3 x) + \frac{d}{dx}(\cos^6 x)$$

$$= 3\sin^2 x \times \frac{d}{dx}(\sin x) + 6\cos^5 x \frac{d}{dx}(\cos x)$$

$$= 3\sin^2 x \times \cos x + 6\cos^5 x(-\sin x)$$

$$= 3\sin^2 x\cos x(\sin x - 2\cos^4 x)$$

3.

$\mathbf{y = (5x)^{3\cos 2x}}$.

Ans - Given function is $y = (5x)^{3\cos 2x}$

Taking logarithm on both sides of the function.

$\log y = 3\cos 2x \log 5x$

Differentiating both sides w.r.t. x gives

$$\frac{1}{y}\frac{dy}{dx} = 3\left[\log 5 \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\log 5x(-\sin 2x) \cdot \frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x} \cdot \cdot \frac{d}{dx}(5x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[-2\sin 2x\log 5x + \frac{\cos 2x}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3y\left[\frac{3\cos 2x}{x} - 6\sin 2x\log 5x\right]$$

$$\therefore \frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x\log 5x\right]$$

4.

$\mathbf{y = \sin^{-1}(x\sqrt{x}), 0 \le x \le 1.}$

Ans - Given function is $y = \sin^{-1}(x\sqrt{x})$

Differentiating both sides w.r.t. x by using chain rule gives

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}(x\sqrt{x})$$

$$= \frac{1}{\sqrt{1-(x\sqrt{x})^2}} \times \frac{d}{dx}(x\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x^3}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2} \cdot x\frac{1}{2}$$

$$= \frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

5.

$$\mathbf{y = \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2}$$

Ans - Given function is

$$y = \frac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$$

Differentiating both sides w.r.t. x using quotient rule gives

$$\frac{dy}{dx} = \frac{\sqrt{2x+7}\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right) - \left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}(\sqrt{2x+7})}{(\sqrt{2x+7})^2}$$

$$= \frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{2}\right)\right] - \left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\cdot\frac{d}{dx}(2x+7)}{\sqrt{2x+7}}$$

$$= \frac{\sqrt{2x+7}\frac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)} - \frac{\cos^{-1}\frac{x}{2}}{(\sqrt{2x+7})(2x+7)}$$

$$\therefore \frac{dy}{dx} = -\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}} + \frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

6.

$$\mathbf{y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < 2}$$

Ans - Given function is

$$y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right] \quad \ldots\ldots(1)$$

Now,

$$\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$$

$$= \frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$= \frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)-(1-\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$= \frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$= \frac{1+\cos x}{\sin x}$$

$$= \frac{2\cos^2\frac{x}{2}}{2\sin x\frac{x}{2}\cos\frac{x}{2}}$$

$$\Rightarrow \frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}} = \cot\frac{x}{2} \quad \ldots\ldots(2)$$

So, from the equations (1) and (2) we obtain,

$$y = \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y = \frac{x}{2}$$

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(x)$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

7.

$$\mathbf{y = (\log x)^{\log x}, x > 1}$$

Ans - Given function is $y = (\log x)^{\log x}$

First, taking logarithm on both sides of the function

$$\log y = \log x \times \log(\log x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\log x \times \log(\log x)]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x) \times \frac{d}{dx}(\log x) + \frac{d}{dx}[\log(\log x)]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x) \times \frac{1}{x} + \log x \times \frac{1}{\log x} \times \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

8.

$y = \cos(a\cos x + b\sin x)$, for some constants a and b.

Ans - Given function is $y = \cos(a\cos x + b\sin x)$

Differentiating both sides w.r.t. x by using chain rule of derivatives gives

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x) \times \frac{d}{dx}(a\cos x + b\sin x)$$

$$= -\sin(a\cos x + b\sin x) \times [a(-\sin x) + b\cos x]$$

$$\therefore \frac{dy}{dx} = (a\sin x - b\cos x) \times \sin(a\cos x + b\sin x)$$

9.

$y = (\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

Ans - Given function is $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

First, take the logarithm on both sides of the function

$$\log y = \log\left[(\sin x - \cos x)^{(\sin x - \cos x)}\right]$$

$$\Rightarrow \log y = (\sin x - \cos x) \times \log(\sin x - \cos x)$$

Now, differentiating both sides w.r.t. x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[(\sin x - \cos x) \times \log(\sin x - \cos x)]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x) \times \frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x) \times \frac{d}{dx}\log(\sin x - \cos x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x) \times (\cos x + \sin x) + (\sin x - \cos x) \times \frac{1}{(\sin x - \cos x)} \times \frac{d}{dx}(\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)(\cos x + \sin x)}[1 + \log(\cos x + \sin x)]$$

10.

$y = x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.

Ans - Given function is $y = x^x + x^a + a^x + a^a$

Now, assume that $x^x = u$,

$x^a = v, a^x = w$ and $a^a = s$

$\Rightarrow y = u + v + w + s.$

Differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \ldots.. \quad (1)$$

Also, $u = x^x$

$\Rightarrow \log\ u = \log\ x^x$

$\Rightarrow \log\ u = x\log\ x$

Differentiating both sides w.r.t. x gives

$$\frac{1}{u}\frac{du}{dx} = \log\ x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log\ x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log\ x.1 + x.\frac{1}{x}\right] \qquad \dots\dots(2)$$

$$\therefore \frac{du}{dx} = x^x[\log\ x + 1] = x^x(1 + \log\ x)$$

Again, $v = x^{\mathbf{a}}$

Differentiating both sides w.r.t. x gives

$$\frac{du}{dx} = \frac{d}{dx}(x^a)$$

$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \qquad \dots..(3)$$

Also, $w = a^x$

$$\Rightarrow \log\ w = \log\ a^x$$

$$\Rightarrow \log\ w = x\log\ a$$

So, differentiating both sides w.r.t. x gives

$$\frac{1}{w}\cdot\frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dw}{dx} = w\log a$$

$$\Rightarrow \frac{dw}{dx} = a^x \log a \qquad \ldots.(4)$$

And $s = a^a$

Then differentiating both sides w.r.t. x gives

$$\frac{ds}{dx} = 0 \qquad \ldots(5)$$

As a is constant, and so a^a is also a constant.

Now, from the equations (1), (2), (3), (4), and (5) we have

$$\frac{dy}{dx} = x^x(1 + \log\ x) + ax^{a-1} + a^x\log\ a + 0$$

$$\therefore \frac{dy}{dx} = x^x(1 + \log\ x) + ax^{a-1} + a^x\log\ a$$

11.

$\mathbf{y = x^{x^2-3} + (x-3)^{x^2}}$**, for** $\mathbf{x > 3}$**.**

Ans - Given function is $y = x^{x^2-3} + (x-3)^{x^2}$

Now suppose that $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$\Rightarrow y = u + v$

Now, differentiating both sides w.r.t. x gives

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \ldots\ldots.(1)$$

Also, $u = x^{x^2-3}$

Take the logarithm on both sides of the equation

$$\Rightarrow \log u = \log(x^{x^2-3})$$

$$\Rightarrow \log u = (x^2-3)\log x$$

Differentiating both sides w.r.t. x gives

$$\frac{1}{u}\frac{du}{dx} = \log x.\frac{d}{dx}(x^2-3) + (x^2-3).\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x.\,2x + (x^2-3).\frac{1}{x}$$

Hence, $\frac{du}{dx} = x^{x^2-3}.\left[\frac{x^2-3}{x} + 2\times\log x\right] \quad \ldots.(2)$

Again, $v = (x-3)^{x^2}$

Take the logarithm on both sides of the equation

$$\frac{1}{u}.\frac{dv}{dx} = \log(x-3).\frac{d}{dx}(x^2) + x^2.\frac{d}{dx}\log(x-3]$$

$$\Rightarrow \frac{1}{u}.\frac{dv}{dx} = \log(x-3).\,2x + x^2.\frac{1}{x-3}.\frac{d}{dx}(x-3)$$

$$\Rightarrow \frac{dv}{dx} = v.\left[2x\log(x-3) + \frac{x^2}{x-3}.1\right]$$

Hence, $\frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right] \quad \ldots.(3)$

Thus, from the equations (1), (2) and (3) we obtain

$$\frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x\log x\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x\log(x-3)\right]$$

12.

Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t), x = 10(t - \sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Ans - Given equations are $y = 12(1 - \cos t)$, ...(1)

and $x = 10(t - \sin t)$ ...(2)

Differentiating the equations (1) and (2) w.r.t. x gives

$$\frac{dx}{dt} = \frac{d}{dt}[10(t - \sin t)] = 10 \times \frac{d}{dt}(t - \sin t) = 10(1 - \cos t)$$

$$\frac{dy}{dt} = \frac{d}{dt}[12(1 - \cos t)] = 12 \times \frac{d}{dt}(1 - \cos t)$$
$$= 12 \times [0 - (-\sin t)] = 12\sin t$$

Therefore, by dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$ we get

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{12\sin t}{10(1 - \cos t)} = \frac{12 \times 2\sin\frac{t}{2} \times \cos\frac{t}{2}}{10 \times 2\sin^2\frac{t}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{6}{5}\cot\frac{t}{2}$$

13.

Find $\frac{dy}{dx}$, if $y = \sin^{-1}x + \sin^{-1}\sqrt{1 - x^2}, 0 < x < 1$.

Ans - Given equation is $y = \sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$

Differentiating both sides of equation w.r.t. x gives

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}x + \sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x) + \frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-1+x^2}} \frac{1}{2 \times \sqrt{1-x^2}} \times 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = 0$$

14.

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ for $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Ans - Given equation is $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow x\sqrt{1+y} = y\sqrt{1+x}$$

Now, squaring both sides of the equation gives

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow x^2 + x^2y = y^2 + xy^2$$

$$\Rightarrow x^2 - y^2 = xy^2 - x^2y$$

$$\Rightarrow x^2 - y^2 = xy(y-x)$$

$$\Rightarrow (x+y)(x-y) = xy(y-x)$$

$$\therefore x + y = -xy$$

$$\Rightarrow (1+x)y = -x$$

$$\Rightarrow y = \frac{-x}{(1+x)}$$

Now, differentiating both sides of the equation w.r.t. x gives

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2} = -\frac{(1+x) - x}{(1+x)^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

15.

If $(x-a)^2 + (y-b)^2 = c^2$, for some constant $c > 0$ prove that $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b.

Ans - Given equation is $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides of the equation w.r.t. x gives

$$\frac{d}{dx} = [(x-a)^2] + \frac{d}{dx}[(y-b)^2] = \frac{d}{dx}(c^2)$$

$$\Rightarrow 2(x-a).\frac{d}{dx}(x-a) + 2(y-b).\frac{d}{dx}(y-b) = 0$$

$$\Rightarrow 2(x-a).1 + 2(y-b).\frac{dy}{dx} = 0$$

Hence, $\frac{dy}{dx} = \frac{-(x-a)}{y-b}$

Again, differentiating both sides of equation w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$= \frac{(y-b).\frac{d}{dx}(x-a) - (x-a).\frac{d}{dx}(y-b)}{(y-b)^2}$$

$$= \left[\frac{(y-b) - (x-a).\frac{dy}{dx}}{(y-b)^2}\right]$$

$$= \frac{(y-b) - (x-a).\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}$$

$$= -\left[\frac{(y-b)^2 + (x+a)^2}{(y-b)^2}\right]$$

Therefore,

$$\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}} = \frac{\left[\left(1+\frac{(x-a)^2}{(y-b)^2}\right)\right]^{\frac{3}{2}}}{-\left[\frac{(y-a)^2+(x-a)^2}{(y-a)^3}\right]}$$

$$= \frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-a)^2+(x-a)^2}{(y-a)^3}\right]} = \frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}$$

$$\Rightarrow \left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right]^{\frac{3}{2}} = \frac{\frac{c^2}{(y-b)^3}}{\frac{c^2}{(y-b)^3}} = -c,$$

is a constant, and is independent of a and b.

16.

If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Ans - Given equation is $\cos y = x\cos(a+y)$

Differentiating both sides of equation w.r.t. x gives

$$\frac{d}{dx}[\cos y] = \frac{d}{dx}x\cos(a+y)$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y).\frac{d}{dx}(x) + x.\frac{d}{dx}\cos(a+y)$$

$$\Rightarrow -\sin y\frac{dy}{dx} = \cos(a+y) + x(\sin(a+y)\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y) - \sin y]\frac{dy}{dx} = \cos(a+y) \ldots (1)$$

Since $\cos y = x\cos(a+y)$

$$\Rightarrow x = \frac{\cos y}{\cos(a+y)}$$

So from the equation (1) gives,

$$\left[\frac{\cos y}{\cos(a+y)}.\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$$

$$\Rightarrow (\cos y.\sin(a+y) - \sin y.\cos(a+y)\frac{dy}{dx} = \cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx} = \cos^2(a+y)$$

Hence, it has been proved that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

17.

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$.

Ans - Given statements are

$x = a(\cos t + t\sin t)$ ... (1)

$y = a(\text{sint} - \text{tcost})$ ... (2)

Differentiating both sides of equation (1) w.r.t. x gives

$$\frac{dx}{dt} = a\left[-\text{sint} + \text{sint}.\frac{d}{dx}(t) + t.\frac{d}{dt}(\text{sint})\right]$$

$= a\,[-\text{sint}+\text{sint}+\text{cost}]=\text{atcost}$

Again, differentiating both sides of eqn (2) w.r.t. x gives

$$\frac{dy}{dt} = a.\frac{d}{dt}(\text{sint} - \text{tcost})$$

$$a\left[\text{cost} - \left\{\text{cost}.\frac{d}{dt}(t) + t.\frac{d}{dt}(\text{cost})\right\}\right]$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dx}\right)} = \frac{\text{atsint}}{\text{atcost}} = \text{tant}$$

Now, differentiating both sides w.r.t. x gives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\text{tant}) = \sec^2 t.\frac{dt}{dx} = \sec^2 t.\frac{1}{\text{atcost}}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\sec^3(t)}{at}$$

18.

If $f(x) = |x^3|$, show that $f''(x)$ exists for all real x and find it.

Ans - Remember that, $|x| = \begin{Bmatrix} x, \text{ifx} \geq 0 \\ -x, \text{ifx0} \end{Bmatrix}$

$\Rightarrow$ if $x \geq 0$,then $f(x) = |x|^3 = x^3$

Then, $\{f\}\ (x) = 3x^2$

Differentiating both sides w.r.t. x gives

$\{\{f\}'\}'\ (x) = 6x$

Now, if x 0, then $f(x) = |x|^3 = (-x^3) = x^3$.

So, $\{f\}'(\mathbf{x}) = 3x^2$

Therefore, differentiating both sides w.r.t. x gives

$\{\{f\}'\}'\ (x) = 6x$

Hence, for $f(x) = |x|^3$, $\{\{f\}'\}\ (\mathbf{x})$ exists for all real values of x and is provided as

$\{\{f\}'\}'(x) = \begin{Bmatrix} 6x, \text{ifx} \geq 0 \\ -6x, \text{ifx0} \end{Bmatrix}$

19.

Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

Ans - Given sum formula is $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Now differentiating both sides w.r.t. x gives

$$\frac{d}{dx}[\sin(A+B)] = \frac{d}{dx}(\sin A \cos B) + \frac{d}{dx}(\cos A \sin B)$$

$$\Rightarrow \cos(A+B) \times \frac{d}{dx}(A+B)$$
$$= \cos B \times \frac{d}{dx}(\sin A) + \sin A \times \frac{d}{dx}(\cos B)$$
$$+ \sin B \times \frac{d}{dx}(\cos A) + \cos A \times \frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B) \times \frac{d}{dx}(A+B)$$

$$= \cos B \times \cos A \frac{d}{dx} + \sin A(-\sin B)\frac{dB}{dx}$$

$$+ \sin B(-\sin A) \times \frac{dA}{dx} + \cos A\cos B \times \frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\left[\frac{dA}{dx} + \frac{dB}{dx}\right]$$

$$= (\cos A\cos B - \sin A\sin B) \times \left[\frac{dA}{dx} + \frac{dB}{dx}\right]$$

Hence, the required sum formula for cosines is
$\cos(A+B) = \cos A\cos B - \sin A\sin B$

20.

Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

Ans - Let take the function $f(x)=|x|+|x-1|$

Observe that, the function f is continuous everywhere, but not differentiable at $x=0$ and $x=1$.

21.

If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$**, prove that**

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Ans - Given function is $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Evaluate the determinant.

$$y = (mc - nb)f(x) - (lc - na)g(x) + (lb - ma)h(x)$$

Now, differentiating both sides w.r.t. x gives

$$\frac{dy}{dx}=\frac{d}{dx}[(mc-nb)f(x)]-\frac{d}{dx}[(lc-na)g(x)]+\frac{d}{dx}[(lb-ma)h(x)]$$

$$=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$

$$=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\therefore \frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

22.

If $y = e^{a\cos^{-1}x}, -1 \le x \le 2$,

show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Ans - Given equation is $y = e^{a\cos^{-1}x}$

Taking logarithm on both sides of the equation.

$\log y = a\cos^{-1}x \log e$

$\Rightarrow \log y = a\cos^{-1}x$

Differentiating both sides w.r.t. x gives

$$\frac{1}{y}\frac{dy}{dx} = ax\frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-ax}{\sqrt{1-x^2}}$$

Therefore, squaring both sides of the equation gives

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2y^2}{1-x^2}$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Again, differentiating both sides w.r.t. x gives

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}(1-x^2) + (1-x^2) \times \frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right] = a^2 \frac{d}{dx}(y^2)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + (1-x^2) \times 2\frac{dy}{dx} \times \frac{d^2y}{dx^2} = a^2 \times 2y \times \frac{dy}{dx}$$

$$\Rightarrow x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = a^2 \times y$$

Hence, it is proved that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$