NCERT Solutions for Class 12 Maths Chapter 11 – Three Dimensional Geometry

Exercise 11.1

1.

If a line makes angles 90°, 135°, 45° with the x, y and z-axes respectively, find its direction cosines.

Ans - $l = \cos 90^\circ = 0$

$$m = \cos 135^\circ = \cos(180^\circ - 45^\circ) = -\cos 45 = -\frac{1}{\sqrt{2}}$$

$$n = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

Hence direction cosines are $0, -\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

2.

Find the direction cosines of a line which makes equal angles with the coordinate axes.

Ans – Let α be the angle. Since linw makes equal angles with the coordinate axes,

$l = m = n = \cos\alpha$

We know, $l^2 + m^2 + n^2 = 1$

$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$3\cos^2\alpha = 1$

$\cos^2\alpha = \frac{1}{3}$

$\cos\alpha = \pm\frac{1}{\sqrt{3}}$

$\therefore$ Direction cosines will be $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$ and $\pm\frac{1}{\sqrt{3}}$

3.

If a line has the direction ratios –18, 12, – 4, then what are its direction cosines ?

Ans - Given direction ratios a = -18, b = 12 and c = -4

$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$

$l = \frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$

$l = -\frac{18}{22}$

$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$

$m = \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$

$m = \frac{12}{22}$

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$

$$n = -\frac{2}{22}$$

Therefore the direction cosines are $-\frac{18}{22}, \frac{12}{22}$ and $-\frac{2}{22}$

4.

Show that the points (2, 3, 4), (– 1, – 2, 1), (5, 8, 7) are collinear.

Ans - Given points are A(2,3,4), B(−1,−2,1) and C(5,8,7).

Direction ratio of line joining the two coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as $(x_2 - x_1)$, $(y_2 - y_1)$ and $(z_2 - z_1)$

⇒ Direction ratio of line AB is given as $(-1-2)$,$(-2-3)$ and $(1-4)$.

⇒ Direction ratio of line AB is -3,-5 and -3 .

Similarly direction ratio of line BC is given as $(5-(-1))$,$(8-(-2))$ and $(7-1)$.

⇒ Direction ratio of line BC is 6, 10 and 6.

On comparing the direction ratio of AB and BC, it can be seen that the direction ratio of BC is -2 times of AB i.e. they are proportional.

$AB = \lambda(BC)$

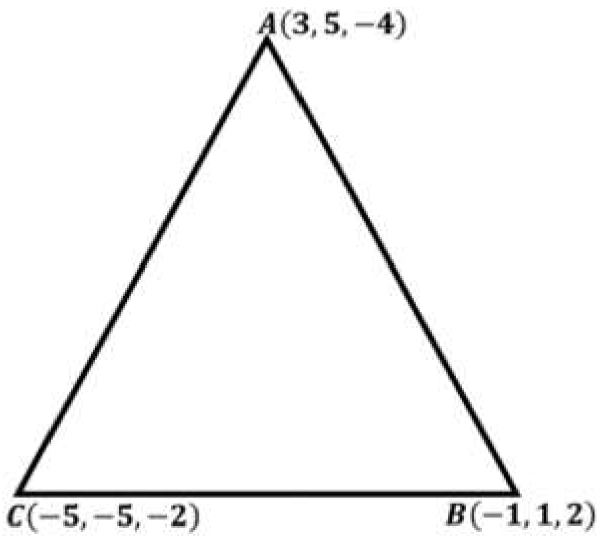
Hence AB∥BC. As point B is common to both AB and BC .

∴ Given points A(2,3,4),B(−1,−2,1) and C(5,8,7) are collinear.

5.

Find the direction cosines of the sides of the triangle whose vertices are (3, 5, – 4), (– 1, 1, 2) and (– 5, – 5, – 2).

Ans - Given vertices of ΔABC are A(3,5,−4),B(−1,1,2) and C(−5,−5,−2).



Direction ratio of line joining the two coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is given as $(x_2 - x_1)$, $(y_2 - y_1)$ and $(z_2 - z_1)$

⇒ Direction ratio of line AB is given as $(-1-3)$, $(1-5)$ and (2 – (–4)) .

⇒ Direction ratio of line AB is −4,−4 and 6.

Direction cosines of side AB using direction ratio are,

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

$$l = \frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}$$

$$l = -\frac{4}{2\sqrt{17}}$$

$$l = -\frac{2}{\sqrt{17}}$$

$$m = \frac{a}{\sqrt{a^2+b^2+c^2}}$$

$$m = \frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}$$

$$m = -\frac{4}{2\sqrt{17}} = -\frac{2}{\sqrt{17}}$$

$$n = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{6}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}$$

$$n = \frac{6}{2\sqrt{17}}$$

$$n = \frac{3}{\sqrt{17}}$$

$\therefore$ Direction cosines of AB are $-\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}$ and $\frac{3}{\sqrt{17}}$

Calculating the direction cosines of side BC

Direction ratio of BC is given as $(-5-(-1)), (-5-1)$ and $(-2-2)$

$\Rightarrow$ Direction ratio of BC is $-4, -6$ and -4.

Direction cosines of side BC using direction ratio are

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}$$

$$l = -\frac{4}{2\sqrt{17}}$$

$$l = -\frac{2}{\sqrt{17}}$$

$$m = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{-6}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}$$

$$m = -\frac{6}{2\sqrt{17}} = -\frac{3}{\sqrt{17}}$$

$$n = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}$$

$$n = -\frac{4}{2\sqrt{17}}$$

$\therefore$ Direction cosines of BC is $-\frac{2}{\sqrt{17}}, -\frac{3}{\sqrt{17}}$ and $-\frac{2}{\sqrt{17}}$.

Calculating the direction cosines of side AC

Direction ratio of AC is given as $(-5-3), (-5-5)$ and $(-2-(-4))$

$\Rightarrow$ Direction ratio of AC is -8, -10 and 2

Direction cosines of side AC using direction ratio is given as

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$l = \frac{-8}{\sqrt{(-8)^2 + (-10)^2 + (2)^2}}$$

$$l = -\frac{8}{2\sqrt{42}}$$

$$l = -\frac{4}{\sqrt{42}}$$

$$m = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{-10}{\sqrt{(-8)^2 + (-10)^2 + (2)^2}}$$

$$m = -\frac{10}{2\sqrt{42}} = -\frac{5}{\sqrt{42}}$$

$$n = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$n = \frac{2}{\sqrt{(-8)^2 + (-10)^2 + (2)^2}}$$

$$n = \frac{2}{2\sqrt{42}} = \frac{1}{\sqrt{42}}$$

$\therefore$ Direction cosines of AC is $-\frac{4}{\sqrt{42}}, -\frac{5}{\sqrt{42}}$ and $\frac{1}{\sqrt{42}}$

Therefore,

Direction cosines of AB is $-\frac{2}{\sqrt{17}}, -\frac{2}{\sqrt{17}}$ and $\frac{3}{\sqrt{17}}$

Direction cosines of BC is $-\frac{2}{\sqrt{17}}, -\frac{3}{\sqrt{17}}$ and $-\frac{2}{\sqrt{17}}$

Direction cosines of AC is $-\frac{4}{\sqrt{42}}, -\frac{5}{\sqrt{42}}$ and $\frac{1}{\sqrt{42}}$.

Exercise 11.2

1.

Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Ans - Two lines with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular to each other if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

(i) For the lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$, we obtain

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \frac{12}{13} \times \frac{4}{13} + \frac{-3}{13} \times \frac{12}{13} + \frac{-4}{13} \times \frac{3}{13}$$

$$= \frac{48}{169} - \frac{36}{169} - \frac{12}{169}$$

$$= 0$$

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines

$\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$, we obtain

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \frac{4}{13} \times \frac{-3}{13} + \frac{12}{13} \times \frac{-4}{13} + \frac{3}{13} \times \frac{12}{13}$$

$$= \frac{12}{169} - \frac{48}{169} + \frac{36}{169}$$

$$= 0$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines

$\frac{3}{12}, \frac{-4}{12}, \frac{12}{12}$ and $\frac{12}{12}, \frac{-3}{12}, \frac{-4}{12}$, we obtain

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \frac{3}{13} \times \frac{12}{13} + \frac{-4}{13} \times \frac{-3}{13} + \frac{12}{13} \times \frac{-4}{13}$$

$$= \frac{36}{169} + \frac{12}{169} - \frac{48}{169}$$

$$= 0$$

Therefore, the lines are perpendicular.

2.

Show that the line through the points $(1, -1, 2)$ and (3, 4, -2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Ans - Let AB be the line joining points, $(1, -1, 2)$ and $(3, 4, -2)$ and CD be the line through points $(0, 3, 2)$ and $(3, 5, 6)$.

The direction ratios a_1, b_1, c_1 of AB are

$(3-1), (4(-1)), (-2-2)$

i.e. $2, 5, -4$

The direction ratios a_2, b_2, c_2 of CD are

$(3-0), (5-3), (6-2)$

i.e. $3, 2, 4$

AB and CD willbe perpendicular to each other if

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$(2 \times 3) + (5 \times 2) + (-4 \times 4)$

$=6 + 10 - 16$

$= 0$

Therefore, AB and CD are perpendicular to each other.

3.

Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points $(-1, -2, 1), (1, 2, 5)$

Ans - Let AB be the line through points (4,7,8) and (2,3,4)

CD be the line through points, (-1, -2, 1) and (1, 2, 5).

Directions ratios a_1, b_1, c_1 ,of AB are

$(2-4), (3-7), (4-8)$

i.e. 2, -4, -4

Direction ratios a_2, b_2, c_2 of CD are

$(1-(-1)), (2-(-2)), (5-1)$

i.e. 2, 4, 4

AB will be parallel to CD, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1 \ldots (1)$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1 \ldots (2)$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1 \ldots (3)$$

From Equations (1), (2) and (3), we get

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = -1$$

Thus, AB is parallel to CD.

4.

Find the equation of the line which passes through point $(1, 2, 3)$ and is parallel to the vector $3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$.

Ans - Given that line passes through the point A (1, 2, 3). Therefore, the position vector through A is

$$\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$

Also, $\vec{b} = 3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a constant

$$\vec{r} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k} + \lambda(3\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$$

5.

Find the equation of the line in vector and in Cartesian form that passes through the point with positive vector $2\hat{\imath} - \hat{\imath} + 4\hat{k}$ and is in the direction

Ans - Given that line passes through the point with positive vector

$\vec{a} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$

$\vec{b} = \hat{\imath} + 2\hat{\imath} - \hat{k}$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\vec{r} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k} + \lambda(\hat{\imath} + 2\hat{\jmath} - \hat{k})$

This is the required equation in the vector form

$\vec{r} = (2 + \lambda)\ \hat{\imath} + (2\lambda - 1)\ \hat{\jmath} + (4 - \lambda)\ \hat{k}$

Eliminating λ , we obtain the Cartesian form of equation as.

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of line in the cartesian form.

6.

Find the Cartesian equation of the line which passes through the point (-2, 4, -5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Ans - Given that line passes through the point (-2, 4, -5) and is parallel to

$$\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$$

Directon ratios of the line $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$ are 3, 5, 6

Required line is parallel $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

Therefore, its direction ratios are $3k, 5k, 6k$ when $k \neq 0$

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction ratios (a, b, c), is given by

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Therefore, the equation of the required line is.

$$\frac{x+2}{3k}=\frac{y-4}{5k}=\frac{z+5}{6k}$$

$$\Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}=k$$

7.

The Catesian equation of line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its vector form.

Ans - Cartesian equation of the line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$

Given line passes through the point (5, -4, 6). Positive vector of this point is

$\vec{a}=5\hat{\imath}-4\hat{\jmath}+6\hat{k}$

Also, the direction ratios of the given line are 3, 7, 2

This means that the line is in the direction ofthe vector $\vec{b}=3\hat{\imath}+7\hat{\jmath}+2\hat{k}$

Itis known that the line which passes through point A and parallel to $\vec{b}$ is given by

$\vec{r}=\vec{a}+\lambda\vec{b}\,\vec{r}=\left(5\hat{\imath}-4\hat{\jmath}+6\hat{k}\right)+\lambda\left(3\hat{\imath}+7\hat{\jmath}+2\hat{k}\right).$

This is required equation of the given line in vector form.

8.

Find the angle between the following pairs of lines

(i) $\vec{r}=\left(2\hat{\imath}-5\hat{\jmath}+1\hat{k}\right)+\lambda\left(3\hat{\imath}-2\hat{\jmath}+6\hat{k}\right)$ **and** $\vec{r}=\left(7\hat{\imath}-6\hat{k}\right)+\mu\left(\hat{\imath}+2\hat{\jmath}+2\hat{k}\right)$

(ii) $\vec{r}=\left(3\hat{\imath}+\hat{\jmath}-2\hat{k}\right)+\lambda\left(\hat{\imath}-\hat{\jmath}-2\hat{k}\right)$ **and** $\vec{r}=\left(2\hat{\imath}-\hat{\jmath}-56\hat{k}\right)+\mu\left(3\hat{\imath}-5\hat{\jmath}-4\hat{k}\right)$

Ans - ($\boldsymbol{i}$) Let Q be the angle between the given lines.

Angle between the given pairs of lines is given by,

$$\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| \cdot |\vec{b}_2|}\right|$$

Given lines are parallel to the vectors, $\vec{b}_1 = 3\hat{\imath} - 2\hat{\jmath} + 6\bar{k}$
and $\vec{b}_2 = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$, respectively

$$|b_1| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$|b_2| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

$$b_1 \cdot b_2 = \left(3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}\right) \cdot \left(\hat{\imath} + 2\hat{\jmath} + 2\hat{k}\right)$$

$$= 3 \times 1 + 2 \times 2 + 6 \times 2$$

$$= 3 + 4 + 12$$

$$= 19$$

$$\cos Q = \frac{19}{7 \times 3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{19}{21}\right)$$

(ii) Let Q be the angle between the given lines.

Angle the given pairs is given by,

$$\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| \cdot |\vec{b}_2|}\right|$$

The given lines are parallel to the vectors, $\vec{b}_1 = \vec{i} - \vec{j} - 2\vec{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$ respectively.

$$|b_1| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$|b_2| = \sqrt{3^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$b_1 \cdot b_2 = \left(\hat{i} - \hat{j} - 2\hat{k}\right) \cdot \left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$$

$$= 1 \times 3 + (-1) \times (-5) + (-2) \times (-4)$$

$$= 3 + 5 + 8$$

$$= 16$$

$$\cos Q = \frac{16}{\sqrt{6} \times 5\sqrt{2}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

9.

Find the angle between following pairs of lines.

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ **and** $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ **and** $\frac{x-5}{-4} = \frac{y-2}{1} = \frac{z-3}{8}$

Ans - **(i)** Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by,

$\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| \cdot |\vec{b}_2|}\right|.$

The given lines are parallel to the vectors, $\vec{b}_1 = 2\hat{\imath} + 5\hat{\jmath} - 3\hat{k}$ and $\vec{b}_2 = -\hat{\imath} + 8\hat{\jmath} + 4\hat{k}$ respectively.

$|b_1| = \sqrt{2^2 + 5^2 + (-3)^2} = \sqrt{38}$

$|b_2| = \sqrt{(-1)^2 + 8^2 + 4^2} = 9$

$b_1 \cdot b_2 = \left(2\hat{\imath} + 5\hat{\jmath} - 3\hat{k}\right) \cdot \left(-\hat{\imath} + 8\hat{\jmath} + 4\hat{k}\right)$

$= 2 \times (-1) + 5 \times 8 + (-3) \times 4$

$= -2 + 40 - 12$

$= 26$

$\cos Q = \frac{26}{9\sqrt{38}}$

$\Rightarrow Q = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$

(ii) Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by,

$$\cos Q = \left|\frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\cdot|\vec{b_2}|}\right|$$

The given lines are parallel to the vectors, $\vec{b_1} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b_2} = 8\hat{i} + \hat{j} + 8\hat{k}$ respectively.

$$|b_1| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$|b_2| = \sqrt{4^2 + 1^2 + 8^2} = 9$$

$$b_1 \cdot b_2 = \left(2\hat{i} + 2\hat{j} + \hat{k}\right)\cdot\left(4\hat{i} + \hat{j} + 8\hat{k}\right)$$

$$= 2 \times 4 + 2 \times 1 + 1 \times 8$$

$$= 8 + 2 + 8 = 18$$

$$\cos Q = \frac{18}{3 \times 9}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$$

10.

Find the values of p so that the lines

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2} \text{ and } \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are at right angles.

Ans - Given equations can be written in standard form as

$$\frac{x-1}{-3}=\frac{y-2}{2p}=\frac{z-3}{2} \text{ and } \frac{x-1}{3p}=\frac{y-5}{1}=\frac{z-6}{-5}$$

Direction ratios of the lines are

$-3, \frac{2p}{7}, 2$ and

$-\frac{3p}{7}, 1, -5$

Two lines will be perpendicular to each other if.

$a_1a_2+b_1b_2+c_1c_2=0$

$$\left(-3\times\frac{-3p}{7}\right)+\left(\frac{2p}{7}\times 1\right)+\left(2\times(-5)\right)=0$$

$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}-10=0$$

$$\Rightarrow 9p+2p-70=0$$

$$\Rightarrow 11p-70=0$$

$$\Rightarrow 11p=70$$

$$\Rightarrow p=\frac{70}{11}$$

11.

Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

Ans - We are given the equations of lines as

$$\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1} \text{ and } \frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

The direction ratios of the given lines are

$7, -5, 1$ and $1, 2, 3$

Two lines will be perpendicular to each other if

$$a_1a_2+b_1b_2+c_1c_2=0$$

$$7\times 1+(-5)\times 2+1\times 3$$

$$=7-10+3$$

$$=0$$

12.

Find the shortest distance betwee the lines
$\vec{r}=(\hat{\imath}+2\hat{\jmath}+\hat{k})+\lambda(\hat{\imath}-\hat{\jmath}+\hat{k})$ **and**

$\vec{r}=(2\hat{\imath}-\hat{\jmath}-\hat{k})+\mu(2\hat{\imath}+\hat{\jmath}+2\hat{k})$.

Ans - Equation of given lines are,

$\vec{r} = (\hat{\imath} + 2\hat{\jmath} + \hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + \hat{k})$and $\vec{r} = (2\hat{\imath} - \hat{\jmath} - \hat{k}) + \mu(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$

Shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_1 + \mu\vec{b}_2$ is given by

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| \ldots (1)$$

On comparing the equations, we get

$\vec{a}_1 = (\hat{\imath} + 2\hat{\jmath} + \hat{k})$

$\vec{b}_1 = (\hat{\imath} - \hat{\jmath} + \hat{k})$

$\vec{a}_2 = (2\hat{\imath} - \hat{\jmath} - \hat{k})$

$\vec{b}_2 = (2\hat{\imath} + \hat{\jmath} + 2\hat{k})$

$\vec{a}_2 - \vec{a}_1 = (2\hat{\imath} - \hat{\jmath} - \hat{k}) - (\hat{\imath} + 2\hat{\jmath} + \hat{k})$

$= (\hat{\imath} - 3\hat{\jmath} - 2\hat{k})$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= (-2-1)\hat{\imath} - (2-2)\hat{\jmath} + (1+2)\hat{k}$$

$$= -3\hat{\imath} + 3\hat{k}$$

$$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-3)^2 + 3^2}$$

$$= \sqrt{9+9} = \sqrt{18}$$

$$= 3\sqrt{2}$$

Substituting all values obtained in equation (1)

$$d = \left|\frac{(-3\hat{\imath} + 3\hat{k}).(\hat{\imath} - 3\hat{\jmath} - 2\hat{k})}{3\sqrt{2}}\right|$$

$$= \left|\frac{(-3)1 + 3(-2)}{3\sqrt{2}}\right|$$

$$= \left|\frac{-9}{3\sqrt{2}}\right|$$

$$= \frac{3}{\sqrt{2}}$$

13.

Find the shortest distance betwen the lines

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}.$$

Ans – Shortest distance between the linses is given by,

$$d = \left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \right| \quad \dots (1)$$

Comparing the equations, we obtain,

$x_1 = -1, y_1 = -1, z_1 = -1$

$a_1 = 7, b_1 = -6, c_1 = 1$

$x_2 = 3, y_2 = 5, z_2 = 7$

$a_2 = 1, b_2 = -2, c_2 = 1$

Then, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

$= -16 - 36 - 64$

$= -116$

Also

$\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}$

$= \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2}$

$= \sqrt{116} = 2\sqrt{29}$

Substituting all the values in equation(1)

$d = \left| \frac{-116}{2\sqrt{29}} \right|$

$= \left| \frac{-58}{\sqrt{29}} \right| = \left| -2\sqrt{29} \right|$

$= 2\sqrt{29}$

14.

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and}$$
$$\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k}).$$

Ans - The equations of given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{and } \vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k}).$$

It is known that the shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_1 + \mu\vec{b}_2$ is given by

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| \ldots (1)$$

On comparing the equations, we get

$$\vec{a}_1 = (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{b}_1 = (\hat{i} - 3\text{j} + 2\hat{k})$$

$$\vec{a}_2 = (4\hat{i} + 5\hat{j} + 6\hat{k})$$

$$\vec{b}_2 = (2\hat{i} + 3\text{j} + \hat{k})$$

$$\vec{a}_2 - \vec{a}_1 = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= (3\hat{i} + 3\hat{j} + 3\hat{k})$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= (-3 - 6)\hat{i} - (1 - 4)\hat{j} + (3 + 6)\hat{k}$$

$$= -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-9)^2 + 3^2 + 9^2}$$

$$= \sqrt{81 + 9 + 81} = \sqrt{171}$$

$$= 3\sqrt{19}$$

Substituting all values obtained in equation (1)

$$d=\left|\frac{(-9\hat{\imath}+3\hat{\jmath}+9\hat{k})\cdot(3\hat{\imath}+3\hat{\jmath}+3\hat{k})}{3\sqrt{19}}\right|$$

$$=\left|\frac{(-9)3+3\times 3+9\times 3}{3\sqrt{19}}\right|$$

$$=\left|\frac{9}{3\sqrt{19}}\right|$$

$$=\frac{3}{\sqrt{19}}$$

15.

Find the shortest distance between the lines whose vector equations are

$$\vec{r}=(1-t)\hat{\imath}+(t-2)\hat{\jmath}+(3-2t)\hat{k}$$

$$\vec{r}=(s+1)\hat{\imath}+(2s-1)\hat{\jmath}-(2s+1)\hat{k}$$

Ans - Firstly, let's consider the given equations

$$\Rightarrow \vec{r}=(1-t)\hat{\imath}+(t-2)\hat{\jmath}+(3-2t)\hat{k}$$

$$\vec{r}=\hat{\imath}-t\hat{\imath}+t\hat{\jmath}-2\hat{\jmath}+3\hat{k}-2t\hat{k}$$

$$\vec{r}=\hat{\imath}-2\hat{\jmath}+3\hat{k}+t(-\hat{\imath}+\hat{\jmath}-2\hat{k})$$

$$\Rightarrow \vec{r}=(s+1)\hat{\imath}+(2s-1)\hat{\jmath}-(2s+1)\hat{k}$$

$$\vec{r}=(s+1)\hat{\imath}+(2s-1)\hat{\jmath}-(2s+1)\hat{k}$$

$$\vec{r}=s\hat{\imath}+\hat{\imath}+2s\hat{\jmath}-\hat{\jmath}-2s\hat{k}-\hat{\imath}$$

$$\vec{r}=\hat{\imath}-\hat{\jmath}-\hat{k}+s(\hat{\imath}+2\hat{\jmath}-2\hat{k})$$

So, now we need to find the shortest distance between

$\vec{r}=\hat{\imath}-2\hat{\jmath}+3\hat{k}+t(-\hat{\imath}+\hat{\jmath}-2\hat{k})$ and
$\vec{r}=\hat{\imath}-\hat{\jmath}-\hat{k}+s(\hat{\imath}+2\hat{\jmath}-2\hat{k})$

We know that shortest distance between two lines

$\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_1 + \mu\vec{b}_2$

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 \times \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| \ldots (1)$$

By comparing the equations, we get,

$\vec{a}_1 = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}, \vec{b}_1 = -\hat{\imath} + \hat{\jmath} - 2\hat{k}$

$\vec{a}_2 = \hat{\imath} - \hat{\jmath} - \hat{k}, \vec{b}_2 = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$

Since, $(X_1\hat{\imath} + Y_1\hat{\jmath} + Z_1\hat{k}) - (X_2\hat{\imath} + Y_2\hat{\jmath} + Z_2\hat{k})$

$\Rightarrow (X_1 - X_2)\hat{\imath} + (Y_1 - Y_2)\hat{\jmath} + (Z_1 - Z_2)\hat{k}$

So, $\overrightarrow{a_2} - \overrightarrow{a_1} = (\hat{\imath} - \hat{\jmath} - \hat{k}) - (\hat{\imath} - 2\hat{\jmath} + 3\hat{k})$

$= \hat{\jmath} - 4\hat{k} \quad \ldots (2)$

And, $\overrightarrow{b_1} \times \overrightarrow{b_2} = (-\hat{\imath} + \hat{\jmath} - 2\hat{k}) \times (\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$\overrightarrow{b_1} \times \overrightarrow{b_2} = 2\hat{\imath} - 4\hat{\jmath} - 3\hat{k} \ldots (3)$

$\Rightarrow |\overrightarrow{b_1} \times \overrightarrow{b_2}| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{4 + 16 + 9} = 29$

Now multiplying (2) and (3) we get,

$(a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k})\cdot(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}) = a_1a_2 + b_1b_2 + c_1c_2$

$(\overrightarrow{b_1} \times \overrightarrow{b_2})\cdot(\overrightarrow{a_1} \times \overrightarrow{a_2}) = (2\hat{\imath} - 4\hat{\jmath} - 3\hat{k})\cdot(\hat{\jmath} - 4\hat{k})$

$= -4 + 18 = 8 \ \ldots (5)$

By substituting all the values in (1), we get

Shortest distance between the two lines,

$$d = \left|\frac{8}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

$\therefore$ The shortest distance is $\dfrac{8}{\sqrt{29}}$